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GEORGE C. MARSHALL SPACE FLIGHT CENTER

HUNTSVILLE, ALABAMA

THEORY OF FLUID OSCILLATIONS IN PARTIALLY FILLED CYLINDRICAL CONTAINERS

by

Helmut F. Bauer

NASA

ADMINISTRATION

GEORGE C. MARSHALL SPACE FLIGHT CENTER

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ABSTRACT

Liquid in a partially filled container has a strong tendency to "slosh" about even under the slightest disturbance. The mathematical theory for this liquid motion is presented for cylindrical tanks with ring sector cross sections. It is based on linearized potential theory treating the liquid as incompressible, irrotational and non-viscous. Natural frequencies, surface displacement, pressure and velocity distribution in the containers, as well as fluid forces and moments, are presented for various forced vibrations.

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DYNAMICS ANALYSIS BRANCH
AEROBALLISTICS DIVISION

PREFACE

The information contained in this report was presented by Mr. Helmut F. Bauer at a seminar at the Department of Engineering Mechanics, University of Alabama, Tuscaloosa, on November 17, 1961. Mr. Bauer is Chief, Flutter and Vibration Section, Aeroballistics Division, George C. Marshall Space Flight Center, NASA, Huntsville, Alabama, and Associate Professor of Engineering Mechanics at the University of Alabama, Huntsville Center.

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SUMMARY

Liquid in a partially filled container has a strong tendency to "slosh" about even under the slightest disturbance. The mathematical theory for this liquid motion is presented for cylindrical tanks with ring sector cross sections. It is based on linearized potential theory treating the liquid as incompressible, irrotational and non-viscous. Natural frequencies, surface displacement, pressure and velocity distribution in the containers, as well as fluid forces and moments, are presented for various forced vibrations.

I. INTRODUCTION

Fuel sloshing in the tanks of a space vehicle or missile will affect the performance and stability of the vehicle, leading in extreme cases

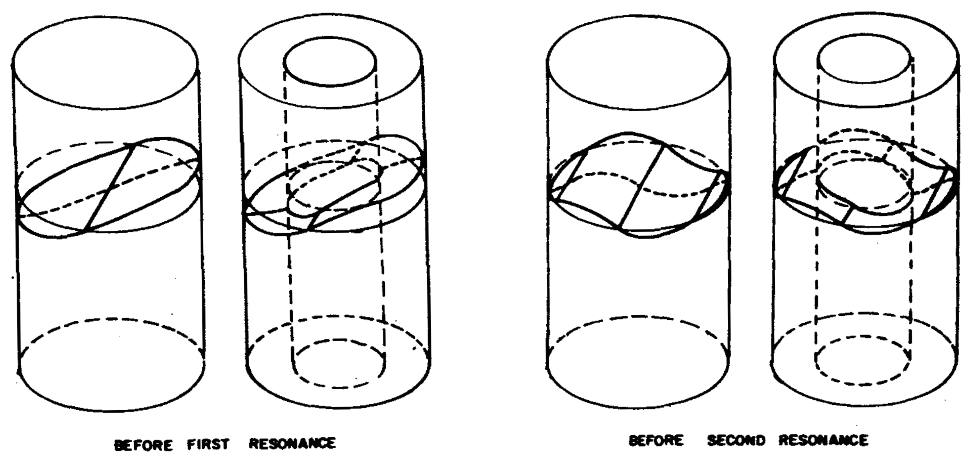

Figure 1. Free Fluid Surface of the Liquid in a Cylindrical Container with Circular and Annular Cross Section

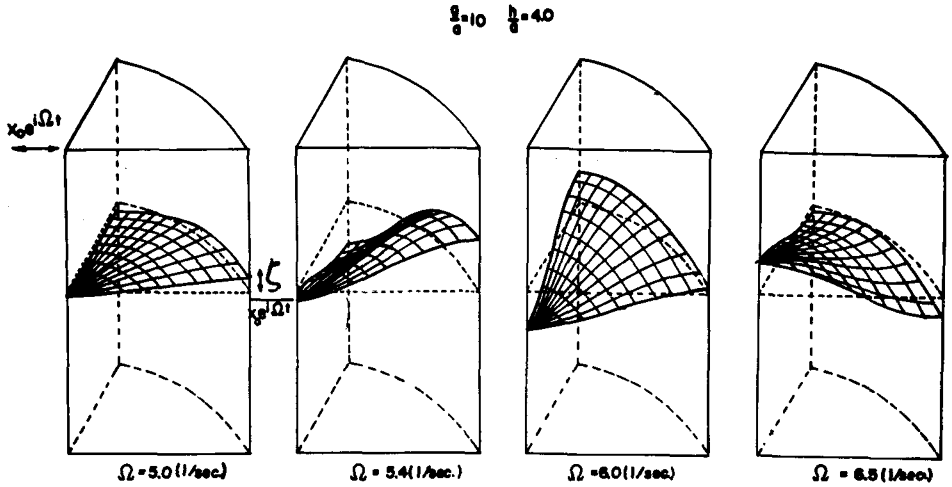

Figure 2. Free Fluid Surface of the Liquid in a Circular Cylindrical Quarter Tank

to catastrophe. Since more than 90% of the total weight of the vehicle at launch is liquid propellant, liquid sloshing represents a major item requiring special attention even in the preliminary design stage of space vehicles. The tendency toward a continuous increase in size of modern space vehicles makes an investigation of this kind even mandatory. If the natural frequencies of the propellant in the tanks are close to the control frequency of the vehicle, the lower modes of the elastic vibration - say the fundamental body-bending modes - or to the natural frequency of the control sensor, the problem obviously becomes acute. Moving propellant exerts forces and moments on the vehicle, which may saturate the control system and thus lead to failure. It is for this reason that, for a realistic dynamic stability and control analysis, the effect of the oscillating propellant has to be considered. "Sloshing" is the term usually applied to a type of liquid motion resulting primarily from translation, pitching, or bending motions of the tank. The free oscillation of a liquid with a free fluid surface as well as the response of that liquid due to these excitations will be treated.

II. DERIVATION OF BASIC EQUATIONS

Since cylindrical tanks with circular sector cross sections are universally used, they are most important. An exact solution of the problem of liquid oscillation with a free fluid surface in a container is practically impossible. For simplification, the liquid is assumed to be incompressible, frictionless, and irrotational. These assumptions are justified since the results describe the dynamic behavior of the liquid very well, as long as the forcing frequency is not too close to the natural frequencies of the liquid system.

The equation of motion of a liquid particle is obtained from the equation of Newton:

$$\frac{d\vec{v}}{dt} = \vec{K} - \frac{1}{\rho} \text{ grad } p \tag{1}$$

where $\vec{K}$ represents the exterior force per unit mass. The acceleration of a liquid particle is

$$\frac{d\vec{v}}{dt} = \left(\frac{\partial}{\partial t} + \vec{v} \cdot \text{grad}\right) \vec{v} = \frac{\partial \vec{v}}{\partial t} + \frac{1}{2} \text{ grad } \vec{v}^{\,2} - [\vec{v} \times \text{curl } \vec{v}] \tag{2}$$

and it is with the acceleration vector $\vec{g}$ as the exterior force per unit mass

$$\frac{\partial \vec{v}}{\partial t} + \frac{1}{2} \text{ grad } \vec{v}^{\,2} - [\vec{v} \times \text{curl } \vec{v}] + \frac{1}{\rho} \text{ grad } p = \vec{g}. \tag{3}$$

These equations represent a system of three non-linear partial differential equations in which the five values $\vec{v} = \{u,v,w\}$, ρ and p are unknown. Since the liquid is considered to be incompressible, the mass density is constant and known. We need only four equations to solve the four unknowns $\vec{v}$ and p. A fourth partial differential equation can be obtained by the principle of mass conservation (continuity equation).

The mass flow through a closed surface without sinks and sources is zero. This can be expressed by the following expression

$$\iint_S \rho\, v_n\, dS = 0$$

with Gauss Theorem

$$\iint_S \rho\, v_n\, dS = \iiint_V \text{div } (\rho \vec{v})\, dV$$

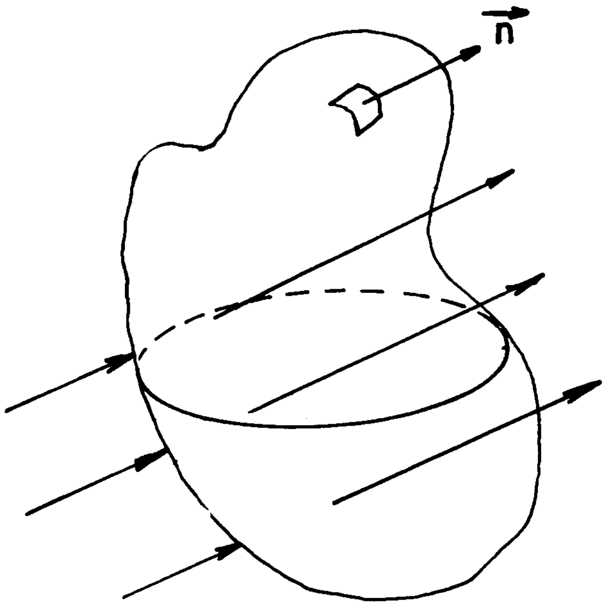

Figure 3. Mass Flow Through a Closed Surface

we obtain for the previous integral

$$\iiint\limits_V \operatorname{div}\,(\rho\vec{v})\; dV = 0$$

for any arbitrary volume. From this we conclude that everywhere $\operatorname{div}(\rho\vec{v}) = 0$. Since the mass density is considered to be constant, we obtain the continuity equation

$$\operatorname{div}\,(\vec{v}) = 0 \tag{4}$$

The equations (3) and (4) are with given boundary and initial conditions sufficient to determine the velocity components u, v, and w and the pressure p uniquely.

Assuming the flow to be irrotational

$$\operatorname{curl}\,\vec{v} = 0$$

the flow field can (due to the identity curl grad $\Phi \equiv 0$) be represented by a velocity potential Φ. From this velocity field, $\vec{v}$ can be obtained as

$$\vec{v} = \text{grad}\ \Phi$$

From this equation and the continuity equation, it can be seen that the velocity potential is a solution of the Laplace equation.

$$\Delta\Phi = 0 \tag{5}$$

This result represents a great simplification in the treatment of the problem since the velocity field can be obtained by one function Φ only, which satifies a linear partial differential equation. Further simplification, which follows from the assumption of the irrotational flow, can be obtained from equation (3) which can be written in the following form:

$$\text{grad}\ (\frac{\partial\Phi}{\partial t}) + \frac{1}{2}\ \text{grad}\ \vec{v}^{2} + \text{grad}\ (\frac{p}{\rho}) = -\ \text{grad}\ (gz) \tag{3a}$$

in which the force per unit mass is in negative z direction and the density is considered to be constant. The integration of this equation leads to the instationary Bernoulli equation of the form

$$\frac{\partial\Phi}{\partial t} + \frac{1}{2}\,v^2 + \frac{p}{\rho} + gz = 0. \tag{6}$$

The potential equation (5) and the Bernoulli equation (6) substitute the continuity equation (4) and the equation of motion (3). Once the potential Φ is obtained, all other values can be determined with it. The velocity distribution is obtained by differentiation with respect to the spacial coordinates of the solution of the differential equation (5). The pressure p is obtained from equation (6). For a fixed or movable boundary T (x,y,z,t) = 0, the boundary conditions can be obtained by setting the normal velocity of the liquid at the boundary equal to the normal velocity of the boundary itself $v_n = v_N$. This results in

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + u\,\frac{\partial T}{\partial x} + v\,\frac{\partial T}{\partial y} + w\,\frac{\partial T}{\partial z} = 0$$

$$\text{or}\quad \frac{dT}{dt} = (\frac{\partial}{\partial t} + \vec{v}\cdot\text{grad})\ T = 0 \tag{7}$$

For fixed time-independent container walls, the boundary condition reads

$$\frac{\partial \Phi}{\partial n} = 0 \qquad \text{at } T.$$

Another boundary with the treatment of liquid oscillations in a partially filled container is that at the free fluid surface. If the equation of such a surface on which the pressure p is given is described by

$$z = \zeta\ (x,y,t),$$

then it is for a fluid particle at the surface

$$T \equiv F = z - \zeta\ (x,y,t) = 0$$

and it is with (T)

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + u\,\frac{\partial F}{\partial x} + v\,\frac{\partial F}{\partial y} + w\,\frac{\partial F}{\partial t} = 0$$

and

$$u = \frac{\partial \Phi}{\partial x}$$

$$v = \frac{\partial \Phi}{\partial y}$$

$$w = \frac{\partial \Phi}{\partial z}$$

at the free surface

$$\frac{\partial \zeta}{\partial t} + \frac{\partial \Phi}{\partial x}\frac{\partial \zeta}{\partial x} + \frac{\partial \Phi}{\partial y}\frac{\partial \zeta}{\partial y} - \frac{\partial \Phi}{\partial z} = 0. \tag{8}$$

Furthermore, we obtain with the Bernoulli equation (6) the pressure at the free surface

$$p = -\rho\ \{\frac{\partial \Phi}{\partial t} + \frac{1}{2}\ [(\frac{\partial \Phi}{\partial x})^2 + (\frac{\partial \Phi}{\partial y})^2 + (\frac{\partial \Phi}{\partial z})^2] + g\zeta\ \}\,. \tag{9}$$

From this we conclude for the pressure p = 0 that

$$\zeta = -\frac{1}{g}\frac{\partial\Phi}{\partial t} - \frac{1}{2g}\{(\frac{\partial\Phi}{\partial x})^2 + (\frac{\partial\Phi}{\partial y})^2 + (\frac{\partial\Phi}{\partial z})^2\}. \tag{10}$$

This is the boundary condition that is obtained with Bernoulli's equation (dynamic condition), while equation (8) represents a kinematic condition. Assuming small velocities and small free surface displacements and derivatives of those, the velocity potential Φ and the free surface displacement ζ can be represented as series

$$\Phi\ (x,y,z,t) = \sum_{\lambda=1}^{\infty} \Phi_\lambda \epsilon^\lambda \tag{11}$$

and

$$\zeta\ (x,y,t) = \sum_{\lambda=0}^{\infty} \zeta_\lambda \epsilon^\lambda \tag{12}$$

where ϵ is a small value.

From this we conclude that Φ_λ is a solution of the Laplace equation

$$\Delta\Phi_\lambda = 0$$

Neglecting second order and higher terms we obtain from equation (10) with equations (11) and (12), that $\zeta_o = 0$ and

$$\zeta_1 = -\frac{1}{g}\frac{\partial\Phi_1}{\partial t} \qquad \text{at } z = \zeta_o = 0$$

and from equation (8) we obtain

$$\frac{\partial\zeta}{\partial t} = \frac{\partial\Phi}{\partial z}$$

Elimination of ζ results in the boundary condition for the free fluid surface:

$$\frac{\partial^2\Phi}{\partial t^2} + g\frac{\partial\Phi}{\partial z} = 0 \tag{13}$$

The displacement of the free fluid surface is

$$\zeta = -\frac{1}{g}\left(\frac{\partial\Phi}{\partial t}\right)_{z=0}. \tag{14}$$

We thus obtain, for the solutions of the incompressible, irrotational and frictionless liquid in a stationary container with free fluid surface, the following equations:

$$\begin{aligned} &\Delta\Phi = 0 \\ &\frac{\partial\Phi}{\partial n} = 0 \qquad \text{at the tank walls} \\ &\frac{\partial^2\Phi}{\partial t^2} + g\frac{\partial\Phi}{\partial z} = 0 \qquad \text{at the free fluid surface.} \end{aligned} \tag{15}$$

The last boundary condition is the only one that has been linearized. For moving containers the basic equations can also be derived in a similar way. Here the boundary conditions of the container walls must also be linearized. The solution of the Laplace equation consists of the potential Φ_0 of the motion of the infinitely long container (which is assumed to be small) and the disturbance potential ψ of the motion of the liquid. The velocity potential Φ can therefore be presented as

$$\Phi = \Phi_0 + \psi. \tag{16}$$

The disturbance potential ψ is due to small translational and rotational container motions which disturb the free fluid surface. We obtain, therefore, the equations for solution of the liquid with a free fluid surface due to forced oscillations of the container.

$$\begin{aligned} &\Delta\Phi = 0 \\ &\left(\frac{\partial\Phi}{\partial n}\right)_{\text{at the container wall}} = \text{normal velocity at the container wall} \end{aligned} \tag{17}$$

$$\frac{\partial^2\Phi}{\partial t^2} + g \frac{\partial\Phi}{\partial t} = 0 \text{ at the free fluid surface}$$

III. FREE OSCILLATIONS

The flow field of a liquid, with free fluid surface in a circular cylindrical ring sector tank with flat tank bottom having a vertex angle $2\Pi\alpha$, is obtained from solution of the Laplace equation with the appropriate linearized boundary conditions.

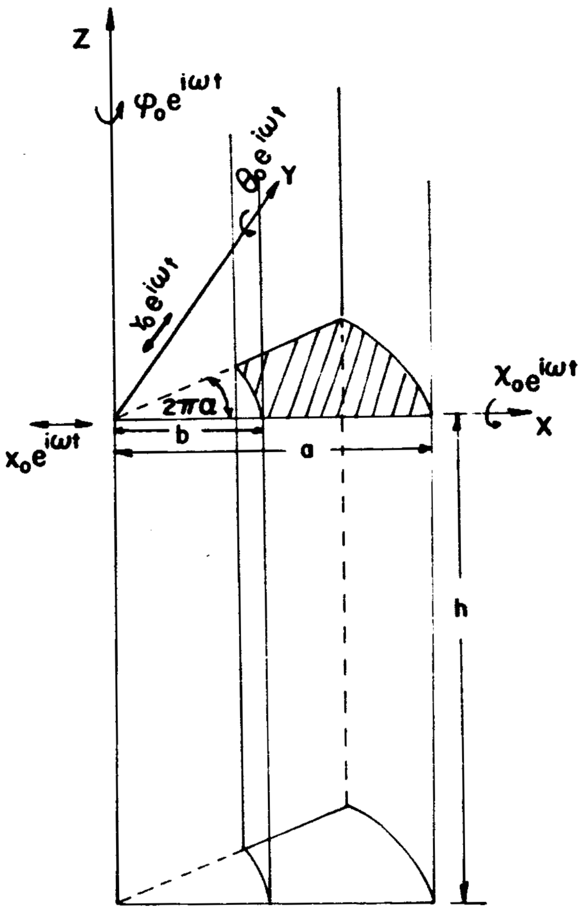

Figure 4.. Tank Geometry

$$\Delta\psi = 0$$

$$\frac{\partial\psi}{\partial z} = 0 \qquad \text{at the tank bottom } z = - h$$

$$\frac{\partial\psi}{\partial r} = 0 \qquad \text{at the circular cylindrical tank walls} \quad r = a,b.$$

$$\frac{1}{r}\frac{\partial\psi}{\partial\varphi} = 0 \qquad \text{at the sector walls} \quad \varphi = 0,\ 2\Pi\alpha \tag{18}$$

$$\frac{\partial^2\psi}{\partial t^2} + g\frac{\partial\psi}{\partial z} = 0 \quad \text{at the free fluid surface} \quad z = 0$$

With Bernoullis separation method in which the function ψ (r,φ,z,t) is written as a product of functions which depend only on one independent variable

$$\psi = R\ (r)\ g\ (\varphi)\ Z\ (z)\ e^{i\omega t} \tag{19}$$

the solution of the Laplace equation can be found. One has to take care in the choice of the separation constants and their signs in order to obtain solutions which describe the physics of the problem.

$$\psi = e^{i\omega t}\ \{C_1 \cos \nu\varphi + C_2 \sin \nu\varphi\}\ [\{C_3 \cosh\lambda z + C_4 \sinh\lambda z\}$$
$$\{C_5 J_\nu (\lambda r) + C_6 Y_\nu (\lambda r)\} + \{C_7 z + C_8\}\ \{C_9 r^\nu + C_{10} r^{-\nu}\}] \tag{20}$$

The velocity potential which satisfies the boundary conditions of the container wall is

$$\psi\ (r,\varphi,z,t) = \sum_m \sum_n A_{mn}\ \ell^{i\omega_{mn} t} \cos\left(\frac{m}{2\alpha}\varphi\right) \frac{\cos h\ [\xi_{mn}(\frac{z}{a} + \frac{h}{a})]}{\cos h\ [\xi_{mn}\frac{h}{a}]} \cdot$$

$$C_{\frac{m}{2\alpha}}\left(\xi_{mn}\frac{r}{a}\right) \tag{21}$$

where the constants A_{mn} are unknown and can be obtained from the initial conditions.

It is

$$C_{\frac{m}{2\alpha}}\left(\xi_{mn}\frac{r}{a}\right) = \begin{vmatrix} J_{\frac{m}{2\alpha}}\left(\xi_{mn}\frac{r}{a}\right) & Y_{\frac{m}{2\alpha}}\left(\xi_{mn}\frac{r}{a}\right) \\ J'_{\frac{m}{2\alpha}}\left(\xi_{mn}\right) & Y'_{\frac{m}{2\alpha}}\left(\xi mn\right) \end{vmatrix} \tag{22}$$

The values ξ_{mn} are the positive roots of the equation

$$\Delta_{\frac{m}{2\alpha}}(\xi) = \begin{vmatrix} J'_{\frac{m}{2\alpha}}(\xi_{mn} k) & Y'_{\frac{m}{2\alpha}}(\xi_{mn} k) \\ J'_{\frac{m}{2\alpha}}(\xi_{mn}) & Y'_{\frac{m}{2\alpha}}(\xi_{mn}) \end{vmatrix} = 0 \tag{23}$$

in which k = b/a represents the diameter ratio of the inner and outer container walls. With the free fluid surface condition, one obtains the natural frequency of the liquid as (Eigen values)

$$\omega^2_{mn} = \frac{g}{a}\, \xi_{mn} \tanh\left(\xi_{mn} \frac{h}{a}\right) \qquad m,n = 0,1,2,\ldots.. \tag{24}$$

It can be seen from the frequency equation that the natural frequency of the propellant increases with the square root of the longitudinal acceleration g and is indirectly proportional to the square root of the tank diameter. For constant longitudinal accelerations and tank dimensions, most of the change in frequency occurs for shallow propellant depths, i.e., for a fluid height of less than one tank diameter for the first mode and even less for higher modes. Due to increasing longitudinal acceleration, the natural frequency of the propellant versus flight time increases. Only during burn out does the fluid height influence overcome the influence of the acceleration g and decrease the frequency again.

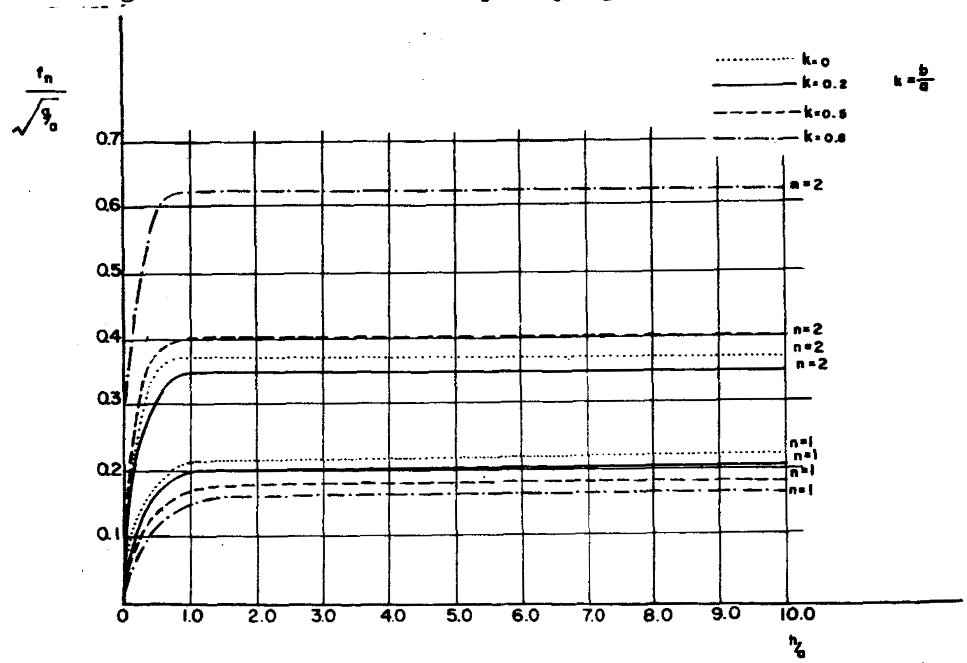

Figure 5. Natural Frequencies of a Liquid in a Cylindrical Container

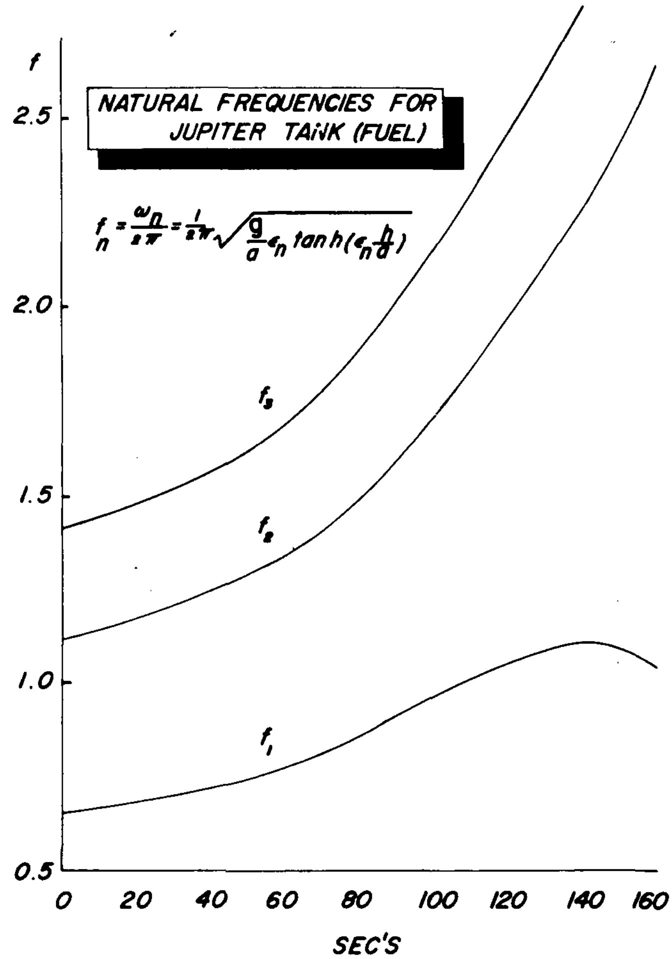

Figure 6. Natural Frequency of Propellant versus Flight Time

IV. FORCED OSCILLATIONS

The flow field of the liquid with the free fluid surface in a circular cylindrical ring sector with a flat bottom, due to forced oscillation of the tank, can again be obtained from the solution of the Laplace equation and the appropriate linearized boundary conditions. In these cases, not only the free fluid surface will be linearized, but also the tank wall conditions have to be presented in a linearized form.

A. Translational Oscillation

For forced oscillations of the tank in the direction of the x-axis, the following set of equations has to be solved:

$$\Delta\Phi = 0$$

$$\frac{\partial\Phi}{\partial z} = 0 \quad \text{for } z = - h$$

$$\frac{1}{r}\frac{\partial\Phi}{\partial\varphi} = 0 \qquad \text{for } \varphi = 0$$

$$\frac{1}{r}\frac{\partial\Phi}{\partial\varphi} = -i\omega x_o\, e^{i\omega t} \sin 2\Pi\alpha \qquad \text{for } \varphi = 2\Pi\alpha \tag{25}$$

$$\frac{\partial\Phi}{\partial r} = i\omega x_o\, e^{i\omega t} \cos\varphi \qquad \text{for } r = a,\, b$$

$$\frac{\partial^2\Phi}{\partial t^2} + g\frac{\partial\Phi}{\partial z} = 0 \qquad \text{for } z = 0$$

By transformation

$$\Phi = \{\psi + i\omega x_o r \cos\varphi\}\, e^{i\omega t}$$

the equation can be written in the form

$$\frac{\partial\psi}{\partial r} = 0 \qquad \text{for } r = a,\, b$$

$$\frac{\partial\psi}{\partial z} = 0 \qquad \text{for } z = -h$$

$$\frac{1}{r}\frac{\partial\psi}{\partial r} = 0 \qquad \text{for } \varphi = 0,\, 2\Pi\alpha \tag{25a}$$

$$g\frac{\partial\psi}{\partial z} - \omega^2\psi = i\omega^3 x_o\, r\cos\varphi \qquad \text{for } z = 0.$$

The solution of the Laplace equation is, with respect to the boundary conditions at the tank walls, given by [(See (21)]

$$\psi\,(r,\varphi,z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{mn}\cos\left(\frac{m}{2\alpha}\varphi\right) C_{\frac{m}{2\alpha}}\left(\xi_{mn}\frac{r}{a}\right)\frac{\cosh\left[\xi_{mn}\left(\frac{z}{a}+\frac{h}{a}\right)\right]}{\cosh\left[\xi_{mn}\frac{h}{a}\right]} \tag{26}$$

With the free fluid surface condition at z = 0, we obtain

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{mn}\cos\left(\frac{m}{2\alpha}\varphi\right) C_{\frac{m}{2\alpha}}\left(\xi_{mn}\frac{r}{a}\right)\left[\frac{g}{a}\xi_{mn}\tanh\left(\xi_{mn}\frac{h}{a}\right) - \omega^2\right] = i\omega^3 r x_o\cos\varphi$$

in which

$$\omega^2_{mn} = \frac{g}{a}\,\xi_{mn}\,\tanh\,(\xi_{mn}\frac{h}{a})$$

represents the square of the natural circular frequency of the system. To determine the unknown coefficients A_{mn}, one has to expand at the right hand side of the previous equation cos φ into a Fourier series and r into a Bessel-Fourier series. The cos φ can be represented as

$$\cos\,\varphi = \sum_{m=0}^{\infty} a_m \cos\,(\frac{m}{2\alpha}\,\varphi)$$

with

$$a_o = \frac{\sin\,2\Pi\alpha}{2\Pi\alpha}$$

$$a_m = \frac{4\alpha\,(-1)^m\,\sin\,2\Pi\alpha}{\Pi\,(m^2 - 4\alpha^2)}$$

The radius r should be represented in a series of the form

$$r = \sum_{n=0}^{\infty} b_{mn}\,C_{\frac{m}{2\alpha}}\,(\xi_{mn}\frac{r}{a}) \qquad m = 0,1,2,.....$$

in which the coefficients are given by

$$b_{mn} = \frac{\int_b^a r^2\,C_{\frac{m}{2\alpha}}\,(\xi_{mn}\frac{r}{a})\,dr}{\int_b^a r\,C^2_{\frac{m}{2\alpha}}\,(\xi_{mn}\frac{r}{a})\,dr}\,.$$

The evaluation of this Bessel integral results in

$$\int_b^a r\, C^2_{\frac{m}{2\alpha}} (\xi_{mn} \frac{r}{a}) \, dr = \frac{a^2}{2} \left[\frac{4}{\pi^2 \xi^2_{mn}} - k^2 C^2_{\frac{m}{2\alpha}} (k\, \xi_{mn})\right]$$

$$- \frac{m^2 a^2}{8\alpha^2 \xi^2_{mn}} \left[\frac{4}{\pi^2 \xi^2_{mn}} - C^2_{\frac{m}{2\alpha}} (k\xi_{mn})\right]$$

$$\int_b^a r^2\, C_{\frac{m}{2\alpha}} (\xi_{mn} \frac{r}{a}) \, dr = a^3 N_2^{(\frac{m}{2\alpha})} (\xi_{mn}).$$

N_2 is represented in the appendix for integer and non-integer values of $m/2\alpha$. The coefficients A_{mn} are therefore

$$A_{mn} = \frac{i\omega \; a_m \; b_{mn} \; x_o}{(\omega^2_{mn}/\omega^2 - 1)}$$

and the velocity potential for translatory excitation of the container in the direction of the x-axis is

$$\Phi (r,\varphi,z,t) = i\omega \, x_o \, e^{i\omega t} \left\{r \cos \varphi + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{a_m \, b_{mn} \, C_{\frac{m}{2\alpha}} (\xi_{mn} \frac{r}{a}) \cos (\frac{m}{2\alpha} \varphi)}{(\frac{\omega^2_{mn}}{\omega^2} - 1)} \cdot \frac{\cosh [\xi_{mn} (\frac{z}{a} + \frac{h}{a})]}{\cosh (\xi_{mn} \frac{h}{a})}\right\}. \tag{27}$$

The first expression in front of the double summation satisfies the boundary conditions at the tank walls while the terms in the double summation vanish at the tank walls. The term $r \cos \varphi$, together with the double summation, satisfies the free boundary condition if one considers the results for the natural frequencies and the representation of $\cos \varphi$ as Fourier series and of r as Bessel-Fourier series. With this obtained velocity potential, one can determine the free surface displacement, the pressure and velocity distribution, and the forces and moments of the liquid by differentiation and integration with respect to the time and spacial coordinates. The free surface displacement of the liquid which is measured from its undisturbed position is

$$\zeta = \frac{\omega^2}{g} x_o e^{i\omega t} \{r \cos\varphi + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{a_m b_{mn}}{(\frac{\omega^2_{mn}}{\omega^2} - 1)} C_{\frac{m}{2\alpha}} (\xi_{mn} \frac{r}{a}) \cos (\frac{m}{2\alpha} \varphi)\}. \quad (28)$$

The pressure distribution at a depth -z in the tank is

$$p = -\rho \frac{\partial \Phi}{\partial t} - \rho g z$$

$$p = \rho\, \omega^2 x_o e^{i\omega t} [r \cos\varphi + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{a_m b_{mn} \cosh [\xi_{mn} (\frac{z}{a} + \frac{h}{a})]}{(\frac{\omega^2_{mn}}{\omega^2} - 1) \cosh (\xi_{mn} \frac{h}{a})}$$

$$C_{\frac{m}{2\alpha}} (\xi_{mn} \frac{r}{a}) \cos (\frac{m}{2\alpha} \varphi)] - \rho g z \quad (29)$$

From the pressure distribution, we obtain by integration the components of the liquid forces and moments, the component of the force in x direction is

$$F_x = \int_0^{2\pi\alpha} \int_{-h}^{o} (a\, p_a - b\, p_b) \cos\varphi d\varphi\, dz - \int_b^a \int_{-h}^{o} p_{\varphi=2\pi\alpha} \sin 2\pi\alpha\, dr\, dz \quad (30)$$

The first term represents the force due to the pressure components at the circular walls, while the second integral is the force component due to the pressure component at the sector-wall $\varphi = 2\pi\alpha$.

This is with the mass of liquid $m = \rho\pi\alpha\, a^2(1 - k^2)\, h$

$$F_x = m\omega^2 x_o e^{i\omega t} [1 + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^{m+1} a_m b_{mn} \sin 2\pi\alpha}{\pi\alpha a (1 - k^2) (\frac{\omega^2_{mn}}{\omega^2} - 1)} \cdot \frac{\tanh (\xi_{mn} \frac{h}{a})}{\xi_{mn} \frac{h}{a}} \cdot$$

$$\{N_o^{(\frac{m}{2\alpha})} (\xi_{mn}) + \frac{4\alpha^2}{(m^2 - 4\alpha^2)} (\frac{2}{\pi\xi_{mn}} - k C_{\frac{m}{2\alpha}} (k\xi_{mn}))\}] \quad (31)$$

the force component in y direction is with the first integral as the force component due to the pressure at the circular walls, the integral due to the pressure at the sector-wall $\varphi = 0$ and the third integral due to the pressure at the sector-wall $\varphi = 2\pi\alpha$

$$F_y = \int_0^{2\pi\alpha}\int_{-h}^{o} (ap_a - bp_b) \sin \varphi d \varphi\, dz - \int_b^a\int_{-h}^{o} p_{\varphi=0}\, drdz + \int_b^a\int_{-h}^{o} p_{\varphi=2\pi\alpha} \cos 2\pi\alpha\, drdz \tag{32}$$

which finally can be represented as

$$F_y = - m\omega^2 e^{i\omega t} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{a_m b_{mn} [1 - (-1)^m \cos 2\pi\alpha]}{\pi\alpha a\, (1 - k^2)} \frac{\tanh (\xi_{mn} \frac{h}{a})}{\xi_{mn} \frac{h}{a}} \cdot \{N_o^{(\frac{m}{2\alpha})} (\xi_{mn}) + \frac{4\alpha^2}{(m^2 - 4\alpha^2)} (\frac{2}{\pi\xi_{mn}} - kC_{\frac{m}{2\alpha}} (\xi_{mn} k))\} \tag{33}$$

Here N_o is

$$N_o^{(\frac{m}{2\alpha})} (\xi_{mn}) = \frac{1}{a} \int_b^a C_{\frac{m}{2\alpha}} (\xi_{mn} \frac{r}{a})\, dr$$

It can be seen that the term in front of the double summation in the x-component of the force represents nothing but the inertial force. The moments of the liquid with respect to the point (0,0,-h/2) are given by

$$M_y = \int_0^{2\pi\alpha}\int_{-h}^{o} (ap_a - bp_b) (\frac{h}{2} + z) \cos \varphi d \varphi\, dz + \int_0^{2\pi\alpha}\int_b^{a} p_c\, r^2 \cos \varphi d \varphi\, dr - \int_b^a\int_{-h}^{o} p_{\varphi=2\pi\alpha} \sin 2\pi\alpha\, (\frac{h}{2} + z)\, drdz \tag{34}$$

where p_c represents the bottom pressure.

The first integral represents the moment due to the pressure at the circular walls, while the second integral is nothing but the bottom pressure's contribution to the moment. The third integral finally is the moment due to the pressure of the sector-wall $\varphi = 2\pi\alpha$.

Similarly, we obtain for the moment about the x-axis

$$M_x = -\int_0^{2\pi\alpha}\int_{-h}^{0} (ap_a - bp_b)\left(\frac{h}{2} + z\right)\sin\varphi d\varphi\, dz - \int_0^{2\pi\alpha}\int_b^{a} p_c\, r^2 \sin\varphi d\varphi\, dr +$$

$$+\int_b^{a}\int_{-h}^{0}\left(\frac{h}{2} + z\right) p_{\varphi=0}\, drdz - \int_b^{a}\int_{-h}^{0}\left(\frac{h}{2} + z\right) p_{\varphi=2\pi\alpha}\cos 2\pi\alpha\, drdz \tag{35}$$

Where M_y represents the moment about an axis passing through the point $(0,0,-h/2)$ parallel to the y-axis, while M_x represents the moment about a straight line parallel to the x-axis through the same point.

The moments are given by

$$M_y = m\omega^2 a\, x_o\, e^{i\omega t}\left[\frac{(1 + k^2)}{4\frac{h}{a}}\left(1 + \frac{\sin 2\pi\alpha\cos 2\pi\alpha}{2\pi\alpha}\right) +\right.$$

$$+\frac{\sin 2\pi\alpha}{2\pi\alpha\, a}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{m+1}\, a_m\, b_{mn}}{\left(\frac{\omega^2_{mn}}{\omega^2} - 1\right)(1 - k^2)\,\xi_{mn}}\cdot\left\{\left[\tanh\left(\xi_{mn}\frac{h}{a}\right) +\right.\right.$$

$$+\frac{2}{\xi_{mn}\frac{h}{a}}\left(\frac{1}{\cosh\left(\xi_{mn}\frac{h}{a}\right)} - 1\right)\Big]\Big[N_o^{\left(\frac{m}{2\alpha}\right)}(\xi_{mn}) + \frac{4\alpha^2}{(m^2 - 4\alpha^2)}\left(\frac{2}{\pi\xi_{mn}} -\right.$$

$$\left. - kC_{\frac{m}{2\alpha}}(\xi_{mn} k)\right)\Big] + \frac{8\alpha^2\,\xi^2_{mn}\, N_2^{\left(\frac{m}{2\alpha}\right)}(\xi_{mn})}{(m^2 - 4\alpha^2)\left(\xi_{mn}\frac{h}{a}\right)\cosh\left(\xi_{mn}\frac{h}{a}\right)}\Big\}\Big] +$$

$$+ mg\frac{a}{3}\frac{(1 - k^3)\sin 2\pi\alpha}{(1 - k^2)\,\pi\alpha} \tag{36}$$

$$M_x = - m\omega^2 a\, x_o\, e^{i\omega t} \left[\frac{(1+k^2)}{4\frac{h}{a}} \frac{\sin^2 2\pi\alpha}{2\pi\alpha} - \frac{1}{2\pi\alpha\, a} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\right.$$

$$\frac{a_m\, b_{mn}\, [1 - (-1)^m \cos 2\pi\alpha]}{(\frac{\omega_{mn}^2}{\omega^2} - 1)\,(1 - k^2)\,\xi_{mn}} \cdot \{[\tanh(\xi_{mn}\frac{h}{a}) + \frac{2}{\xi_{mn}\frac{h}{a}}(\frac{1}{\cosh(\xi_{mn}\frac{h}{a})} - 1)]$$

$$[N_o^{(\frac{m}{2\alpha})}(\xi_{mn}) + \frac{4\alpha^2}{(m^2 - 4\alpha^2)}(\frac{2}{\pi\xi_{mn}} - kC_{\frac{m}{2\alpha}}(\xi_{mn} k)] + \tag{37}$$

$$\left. + \frac{8\alpha^2\,\xi_{mn}^2\, N_2^{(\frac{m}{2\alpha})}(\xi_{mn})}{(m^2 - 4\alpha^2)(\xi_{mn}\frac{h}{a})\cosh(\xi_{mn}\frac{h}{a})}\}\right] + mg\,\frac{a}{3}\,\frac{[1 - \cos 2\pi\alpha]\,(1 - k^3)}{\pi\alpha\,(1 - k^2)}$$

The last term in these expressions represents the moment of the undisturbed liquid about the point (0,0,-h/2). The velocity distribution is obtained by differentiating the velocity potential with respect to the proper coordinates and is:

$$u_r = \frac{\partial\Phi}{\partial r} = i\omega\, x_o\, e^{i\omega t}\left[\cos\varphi + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{a_m\, b_{mn}\cosh[\xi_{mn}(\frac{z}{a} + \frac{h}{a})]}{(\frac{\omega_{mn}^2}{\omega^2} - 1)\cosh(\xi_{mn}\frac{h}{a})\, a}\right.$$

$$\left. C'_{\frac{m}{2\alpha}}(\xi_{mn}\frac{r}{a})\cos(\frac{m}{2\alpha}\varphi)\right]$$

$$u_\varphi = \frac{1}{r}\frac{\partial\Phi}{\partial\varphi} = - i\omega\, x_o\, e^{i\omega t}\left[\sin\varphi + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{a_m\, b_{mn}\,(\frac{m}{2\alpha})\cosh[\xi_{mn}(\frac{z}{a} + \frac{h}{a})]}{(\frac{\omega_{mn}^2}{\omega^2} - 1)\cosh(\xi_{mn}\frac{h}{a})}\right.$$

$$\left. \frac{C_{\frac{m}{2\alpha}}(\xi_{mn}\frac{r}{a})}{r}\sin(\frac{m}{2\alpha}\varphi)\right] \tag{38}$$

$$\omega = \frac{\partial\Phi}{\partial z} = i\omega\, x_o\, e^{i\omega t} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{a_m\, b_{mn}\, \xi_{mn} \sinh\left[\xi_{mn}\left(\frac{z}{a} + \frac{h}{a}\right)\right]}{a\left(\frac{\omega_{mn}^2}{\omega^2} - 1\right)\cosh\left(\xi_{mn}\frac{h}{a}\right)}$$

$$C_{\frac{m}{2\alpha}}\left(\xi_{mn}\frac{r}{a}\right)\cos\left(\frac{m}{2\alpha}\varphi\right)$$

The velocity distribution in the tank can be obtained from these by omitting the first term in the parentheses. That is, for the radial velocity component u_r, we have to omit the term $\cos\varphi$, while for the angular velocity component u_φ, the term $\sin\varphi$ is left out. These terms represent with the coefficients in front of the parentheses nothing but the tank motion itself. Similar results can be obtained for an excitation along the y-axis. In this case, $\sin\varphi$ appears and has to be expanded into Fourier series.

B. Rotational Oscillations

For rotational excitation of the container about one of the coordinate axes, which is now taken at the center of the sector axis in the middle between tank bottom and undisturbed fluid surface, the flow field can be obtained from the solution of the Laplace equation and the appropriate linearized boundary conditions. The boundary conditions for forced rotational oscillations of the container about the y-axis are in linearized form

$$
\begin{aligned}
&\frac{\partial\Phi}{\partial r} = -\, i\omega\,\theta_o\, e^{i\omega t}\, z\cos\varphi && \text{at the tank walls } (r = a,b)\\
&\frac{\partial\Phi}{\partial z} = i\omega\,\theta_o\, e^{i\omega t}\, r\cos\varphi && \text{at the tank bottom } \left(z = -\frac{h}{z}\right)\\
&\frac{1}{r}\frac{\partial\Phi}{\partial\varphi} = 0 && \text{at the tank sector wall } (\varphi = 0) \qquad (39)\\
&\frac{1}{r}\frac{\partial\Phi}{\partial\varphi} = i\omega\,\theta_o\, e^{i\omega t}\, z\sin 2\pi\alpha && \text{at the tank sector wall } (\varphi = 2\pi\alpha)\\
&\frac{\partial^2\Phi}{\partial t^2} + g\frac{\partial\Phi}{\partial z} = 0 && \text{at the free fluid surface } \left(z = +\frac{h}{z}\right)
\end{aligned}
$$

where θ_o is the amplitude of the exciting function. With the transformation:

$$\emptyset = [\psi - i\omega r z\, \theta_o \cos\varphi]\, e^{i\omega t}$$

the boundary conditions at the tank sidewalls ($r = a, b, \varphi = 0, 2\pi\alpha$) can be made homogeneous. Solving now the Laplace equation

$$\Delta\psi = 0,$$

a solution can be found which satisfies the homogeneous boundary conditions at those walls. The unknown coefficient A_{mn} and B_{mn} in this solution can be obtained by satisfying the remaining two boundary conditions at the tank bottom and at the free fluid surface, making use of the previous series expansion for $\cos\varphi$, $\sin\varphi$ and r. From this the velocity potential can finally be obtained as

$$\Phi(r,\varphi,z,t) = i\omega\, e^{i\omega t}\, \theta_o \Big[-rz\cos\varphi + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \Big(A_{mn}\cosh\big(\xi_{mn}\tfrac{z}{a}\big) + B_{mn}\sinh\big(\xi_{mn}\tfrac{z}{a}\big)\Big)\, C_{\frac{m}{2\alpha}}\big(\xi_{mn}\tfrac{r}{a}\big)\cos\big(\tfrac{m}{2\alpha}\varphi\big)\Big] \tag{40}$$

where

$$A_{mn} = \frac{ab_{mn}}{\left(\frac{\omega_{mn}^2}{\omega^2} - 1\right)\cosh\left(\xi_{mn}\frac{h}{a}\right)} \left[\frac{2}{\xi_{mn}}\sinh\left(\frac{\xi_{mn}}{2}\frac{h}{a}\right) - \left(\frac{h}{2a} + \frac{g}{a\omega^2}\right)\cosh\left(\frac{\xi_{mn}}{2}\frac{h}{a}\right)\right] a_m$$

$$B_{mn} = \frac{ab_{mn}}{\left(\frac{\omega_{mn}^2}{\omega^2} - 1\right)\cosh\left(\xi_{mn}\frac{h}{a}\right)} \left[\left(\frac{3g}{a\omega^2} - \frac{h}{2a}\right)\sinh\left(\frac{\xi_{mn}}{2}\frac{h}{a}\right) - \frac{2}{\xi_{mn}}\cosh\left(\frac{\xi_{mn}}{2}\frac{h}{a}\right)\right] a_m$$

The expression in front of the double summation satisfies the boundary conditions at the container side walls while the terms under the double summation vanish at these boundary conditions. The double summation, together with the terms in front of it, satisfies the conditions at the tank bottom and free fluid surface. The free fluid surface displacement of the liquid forces and moments, the velocity, and pressure distribution can be obtained from the velocity potential.

C. Roll Oscillations

For the design of the roll control systems, the knowledge of the liquid oscillations in φ direction is very important. For forced excitation of the container about its sector axis (z axis) with the amplitude φ_o, the flow field can again be obtained from the solution of the Laplace equation with appropriate boundary conditions. The origin of the coordinate system again is placed in the undisturbed free fluid surface with the z axis being the sector axis and pointing out of the liquid. The boundary conditions are:

$$\frac{\partial \Phi}{\partial r} = 0 \qquad \text{at the container walls } (r = a,b)$$

$$\frac{\partial \Phi}{\partial z} = 0 \qquad \text{at the container walls } (z = -h)$$

$$\frac{1}{r}\frac{\partial \Phi}{\partial \varphi} = i\omega r\, \varphi_o\, e^{i\omega t} \qquad \text{at the tank sector walls } (\varphi = 0, 2\pi\alpha) \tag{41}$$

$$\frac{\partial^2 \Phi}{\partial t^2} + g\frac{\partial \Phi}{\partial z} = 0 \qquad \text{at the free fluid surface } (z = 0)$$

Since the boundary conditions cannot be satisfied by one potential function as in the previous cases, one represents the velocity potential Φ as the sum of two potentials $G = G(r,\varphi)$ and $F = F(r,\varphi,z)$

$$\Phi(r,\varphi,z,t) = [G(r,\varphi) + F(r,\varphi,z)]\, e^{i\omega t}$$

Both functions satisfy the Laplace equation. The function $G(r,\varphi)$ is determined such that the boundary conditions at the tank walls $r = a,b$ and $\varphi = 0, 2\pi\alpha$ are satisfied. This represents a solution for an infinitely long tank performing roll oscillations. With the function $F(r,\varphi,z)$, the boundary conditions at the tank bottom and at the free fluid surface are satisfied. The boundary conditions for the function G are

$$\frac{\partial G}{\partial r} = 0 \qquad \text{at the tank walls } (r = a,b)$$

$$\frac{\partial G}{\partial \varphi} = i\omega\, \varphi_o\, r^2 \qquad \text{at the sector walls } (\varphi = 0, 2\pi\alpha) \tag{42}$$

Those of the function F are given by:

$$\frac{\partial F}{\partial r} = 0 \qquad \text{at the tank side walls } (r = a,b)$$

$$\frac{\partial F}{\partial \varphi} = 0 \qquad \text{at the sector wall } (\varphi = 0, 2\pi\alpha) \tag{43}$$

$$g \frac{\partial F}{\partial z} - \omega^2 F = \omega^2 G \qquad \text{at the free fluid surface } (z = 0)$$

$$\frac{\partial F}{\partial z} = 0 \qquad \text{at the tank bottom } (z = - h)$$

The solution of the Laplace equation

$$\Delta G = 0$$

can be obtained with the transformation

$$G\,(r,\varphi) = i\omega\,\varphi_o\,r^2\,(\varphi - \pi\alpha) + \psi\,(r,\varphi)$$

(by which the second boundary conditions will be made homogeneous). The first term in $G(r,\varphi)$ represents the potential Φ_o of the rigid infinite cylinder.

From the solution of the thus obtained Poisson equation

$$\Delta\psi = - 4\,i\omega\,\varphi_o\,(\varphi - \pi\alpha)$$

with the boundary conditions

$$\frac{\partial\psi}{\partial r} = - 2i\omega\,\varphi_o\,r\,(\varphi - \pi\alpha) \quad \text{for } r = a,b$$

$$\frac{\partial\psi}{\partial\varphi} = 0 \qquad \text{for } \varphi = 0, 2\pi\alpha$$

we obtain the value $G = G(r,\varphi)$.

The solution which satisfies the last boundary condition in φ is of the form

$$\psi\,(r,\varphi) = \sum_{m=0}^{\infty} R_m\,(r)\,\cos\,(\frac{m}{2\alpha}\,\varphi\,)$$

Introducing this into the partial differential equation, we finally obtain an infinite number of ordinary differential equations for the functions R_m (r). If one expands the function φ on the right hand side into a cos-series

$$\varphi - \pi\alpha = \sum_{m=0}^{\infty} p_m \cos\left(\frac{m}{2\alpha}\varphi\right)$$

with the coefficients

$$p_o = 0$$

$$p_{2m} = 0$$

$$p_{2m-1} = -\frac{8\alpha}{\pi(2m-1)^2}$$

these differential equations are

$$\frac{d^2 R_o}{dr^2} + \frac{1}{r}\frac{dR_o}{dr} = 0$$

$$\frac{d^2 R_{2m}}{dr^2} + \frac{1}{r}\frac{dR_{2m}}{dr} - \frac{m^2}{\alpha^2 r^2} R_{2m} = 0$$

$$\frac{d^2 R_{2m-1}}{dr^2} + \frac{1}{r}\frac{dR_{2m-1}}{dr} - \frac{(2m-1)^2 R_{2m-1}}{4\alpha^2 r^2} = \frac{32\, i\omega\, \varphi_o\, \alpha}{\pi} \cdot \frac{1}{(2m-1)^2}$$

for m = 1,2,.....

The solution of these are for $\alpha \neq 1/4,\ 3/4$ with the boundary condition in r

$$\left[\frac{\partial\psi}{\partial r} = -2\, i\omega\, \varphi_o\, r \sum p_m \cos\left(\frac{m}{2\alpha}\varphi\right)\right]$$

$$R_o\ (r) = 0$$

$$R_{2m}\ (r) = 0$$

$$R_{2m-1}\ (r) = \frac{32\ i\omega\ \varphi_o\ \alpha^2\ a^2}{\pi(2m-1)\ [(2m-1)^2 - 16\ \alpha^2]}\ \{(\frac{r}{a})^{\frac{2m-1}{2\alpha}} \cdot \frac{(1 - k^{\frac{2m-1}{2\alpha}+2})}{(1 - k^{\frac{2m-1}{2\alpha}})} -$$

$$- (\frac{a}{r})^{\frac{2m-1}{2\alpha}}\ \frac{(k^2 - k^{\frac{2m-1}{2\alpha}})\ k^{\frac{2m-1}{2\alpha}}}{(1 - k^{\frac{2m-1}{2\alpha}})} - \frac{4\alpha}{(2m-1)}\ (\frac{r}{a})^2\ \} \qquad (44)$$

finally the solution G (r,φ) is for $\alpha \neq \frac{1}{4}, \frac{3}{4}$

$$G\ (r,\varphi) = i\omega\ \varphi_o\ r^2\ (\varphi - \pi\alpha) + \frac{32\ i\omega\ \varphi_o \alpha^2\ a^2}{\pi} \sum_{m=1}^{\infty} \frac{\cos\ (\frac{2m-1}{2\alpha}\ \varphi)}{(2m-1)\ [(2m-1)^2 - 16\alpha^2]}$$

$$\{ \frac{(\frac{r}{a})^{\frac{2m-1}{2\alpha}}\ (1 - k^{\frac{2m-1}{2\alpha}+2}) - (\frac{a}{r})^{\frac{2m-1}{2\alpha}}\ (k^2 - k^{\frac{2m-1}{2\alpha}})\ k^{\frac{2m-1}{2\alpha}}}{(1 - k^{\frac{2m-1}{2\alpha}})} -$$

$$- \frac{4\alpha}{(2m-1)}\ (\frac{r}{a})^2\ \} \qquad (45)$$

The solution of the equation $\Delta F = 0$ which satisfies the homogeneous boundary conditions at the tank walls is

$$F\ (r,\varphi,z) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{mn}\ \cos\ (\frac{m}{2\alpha}\ \varphi)\ \frac{\cosh\ [\xi_{mn}\ (\frac{z}{a} + \frac{h}{a})]}{\cosh\ (\xi_{mn}\ \frac{h}{a})}\ C_{\frac{m}{2\alpha}}\ (\xi_{mn}\ \frac{r}{a}) \qquad (46)$$

The constants C_{mn} are obtained with the last boundary condition (43).

Introducing the Fourier series for the function φ in the previously obtained function G (r,φ) and satisfying the boundary condition of the free fluid surface:

$$g \frac{\partial F}{\partial z} - \omega^2 F = \omega^2 G \qquad \text{at } z = 0,$$

one obtains with the Bessel Fourier series for

$$\left(\frac{r}{a}\right)^2 = \sum_{n=0}^{\infty} g_{2m-1,n} \, C_{\frac{2m-1}{2\alpha}} \left(\xi_{2m-1,n} \frac{r}{a}\right)$$

$$\left(\frac{a}{r}\right)^2 = \sum_{n=0}^{\infty} h_{2m-1,n} \, C_{\frac{2m-1}{2\alpha}} \left(\xi_{2m-1,n} \frac{r}{a}\right)$$

$$\left(\frac{r}{a}\right)^{\frac{2m-1}{2\alpha}} = \sum_{n=0}^{\infty} \ell_{2m-1,n} \, C_{\frac{2m-1}{2\alpha}} \left(\xi_{2m-1,n} \frac{r}{a}\right)$$

$$\left(\frac{a}{r}\right)^{\frac{2m-1}{2\alpha}} = \sum_{n=0}^{\infty} q_{2m-1,n} \, C_{\frac{2m-1}{2\alpha}} \left(\xi_{2m-1,n} \frac{r}{a}\right)$$

$$\frac{\int_b^a r^3 \, C_{\frac{2m-1}{2\alpha}} \left(\xi_{2m-1,n} \frac{r}{a}\right) dr}{\int_b^a r \, C^2_{\frac{2m-1}{2\alpha}} \left(\xi_{2m-1,n} \frac{r}{a}\right) dr} = a^2 \, g_{2m-1,n}$$

$$\frac{\int_b^a \frac{1}{r} \, C_{\frac{2m-1}{2\alpha}} \left(\xi_{2m-1,n} \frac{r}{a}\right) dr}{\int_b^a r \, C^2_{\frac{2m-1}{2\alpha}} \left(\xi_{2m-1,n} \frac{r}{a}\right) dr} = a^{-2} \, h_{am-1,n}$$

$$\frac{\int_b^a r^{\frac{2m-1}{2\alpha}+1} \, C_{\frac{2m-1}{2\alpha}} \left(\xi_{2m-1,n} \frac{r}{a}\right) dr}{\int_b^a r \, C^2_{\frac{2m-1}{2\alpha}} \left(\xi_{2m-1,n} \frac{r}{a}\right) dr} = a^{\frac{2m-1}{2\alpha}} \, \ell_{2m-1,n}$$

$$\frac{\int_b^a r^{1-\frac{2m-1}{2\alpha}} C_{\frac{2m-1}{2\alpha}}(\xi_{2m-1,n}\frac{r}{a})\,dr}{\int_b^a r\, C^2_{\frac{2m-1}{2\alpha}}(\xi_{2m-1,n}\frac{r}{a})\,dr} = a^{-\frac{2m-1}{2\alpha}} q_{2m-1,n}$$

the constant C_{mn}

$$C_{2mn} = 0 \qquad m = 0,1,2,\ldots\ldots$$

$$C_{2m-1,n} = \frac{32\, i\omega\, \varphi_o\, \alpha^2 a^2}{\pi\,(2m-1)\,[(2m-1)^2 - 16\alpha^2]\,(\frac{\omega^2_{2m-1,n}}{\omega^2} - 1)}$$

$$\{\frac{\ell_{2m-1,n}(1-k^{\frac{2m-1}{2\alpha}+2}) - q_{2m-1,n}(k^2 - k^{\frac{2m-1}{2\alpha}})\, k^{\frac{2m-1}{2\alpha}}}{(1-k^{\frac{2m-1}{2\alpha}})} -$$

$$- \frac{(2m-1)}{4\alpha} q_{2m-1,n}\}$$

Therefore the solution is

$$F(r,\varphi,z) = \frac{32\, i\omega\, \varphi_o\, a^2\, \alpha^2}{\pi} \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \frac{\cos(\frac{2m-1}{2\alpha}\varphi)\, C_{\frac{2m-1}{2\alpha}}(\xi_{2m-1,n}\frac{r}{a})}{(2m-1)\,[(2m-1)^2 - 16\alpha^2]}$$

$$\frac{\cosh(\xi_{2m-1,n}[\frac{z}{a} + \frac{h}{a}])}{(\frac{\omega^2_{2m-1,n}}{\omega^2} - 1)\cosh(\xi_{2m-1,n}\frac{h}{a})} \cdot \{\frac{\ell_{2m-1,n}(1-k^{\frac{2m-1}{2\alpha}+2}) - q_{2m-1,n}\cdot}{}$$

$$\frac{(k^2-k^{\frac{2m-1}{2\alpha}})\, k^{\frac{2m-1}{2\alpha}}}{(1-k^{\frac{2m-1}{2\alpha}})} - \frac{(2m-1)}{4\alpha} g_{2m-1,n}\} \qquad (47)$$

where $\xi_{2m-1,n}$ are the roots of $C'_{\frac{2m-1}{2\alpha}} = 0$

The velocity potential is finally expressed by:

$$\emptyset (r,\varphi,z,t) = i\omega\varphi_o e^{i\omega t} a^2 \{(\frac{r}{a})^2 (\varphi - \pi\alpha) + \frac{32\alpha}{\pi} \sum_{m=1}^{\infty} \frac{\cos (\frac{2m-1}{2\alpha} \varphi)}{[(2m-1)^2 - 16\alpha^2] (2m-1)} \cdot$$

$$[\frac{(\frac{r}{a})^{\frac{2m-1}{2\alpha}} (1-k^{\frac{2m-1}{2\alpha}+2}) - (\frac{r}{a})^{-\frac{2m-1}{2\alpha}} (k^2-k^{\frac{2m-1}{2\alpha}}) k^{\frac{2m-1}{2\alpha}}}{(1-k^{\frac{2m-1}{2\alpha}})} - \frac{4\alpha}{2m-1} (\frac{r}{a})^2] +$$

$$+ \frac{32\alpha^2}{\pi} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{\cos (\frac{2m-1}{2\alpha} \varphi) C_{\frac{2m-1}{2\alpha}} (\xi_{2m-1,n} \frac{r}{a}) \cosh [\xi_{2m-1,n} (\frac{z}{a} + \frac{h}{a})]}{(2m-1) [(2m-1)^2 - 16\alpha^2] [\frac{\omega^2_{2m-1,n}}{\omega^2} - 1] \cosh (\xi_{2m-1,n} \frac{h}{a})}$$

$$[\frac{\ell_{2m-1,n} (1-k^{\frac{2m-1}{2\alpha}+2}) - q_{2m-1,n} (k^2-k^{\frac{2m-1}{2\alpha}}) k^{\frac{2m-1}{2\alpha}}}{(1-k^{\frac{2m-1}{2\alpha}})} - \frac{2m-1}{4\alpha} q_{2m-1,n}]\} \quad (48)$$

The first term satisfies the boundary condition at the sector walls while the infinite series vanishes them term by term. At the boundary condition at the tank walls $r = a$ and $r = b$, the double summation vanishes, while the simple summation vanishes together with the first term. Surface amplitude, liquid forces and moments, velocity distribution, etc., can be obtained in the same manner as the previous cases by differentiation integrating. As already mentioned, the results are only valid if $\alpha \neq 1/4$ and $3/4$. In these cases the non-homogeneous part of the differential equation R_m is in resonance with the Eigen solution for $m = 1$ and $m = 2$. For a cylindrical container with quarter ring cross section

$$R_1 (r) = C_1 r^2 + D_1 r^{-2} + \frac{2i\omega \varphi_o}{\pi} r^2 \ln r$$

the other solutions of the differential equation in R_m are the same. The integration constants C_1 and D_1 can be obtained from the boundary conditions in r. For this tank type the velocity potential, the free fluid surface displacement and the pressure distribution can be obtained by setting $\alpha = 1/4$ and by substituting for the term with the index m = 1 in the simple summation the value

$$\frac{1}{\pi} \{[1 + \frac{2k^4}{1-k^4} \ln k + 2 \ln (\frac{r}{a})] (\frac{r}{a})^2 - \frac{2k^4 \ln k}{1-k^4} (\frac{a}{r})^2 \} \cos 2\varphi$$

and the double summation

$$\frac{1}{\pi} \sum_{n=0}^{\infty} \frac{[2\beta_n + g_n (\frac{2k^4 \ln k}{1-k^4} - 1) + \frac{2k^4 \ln k}{1-k^4} h_n] \cosh [\xi_{2n} (\frac{z}{a} + \frac{h}{a})] C_2 (\xi_{2n} \frac{r}{a})}{(\frac{\omega_{2n}^2}{\omega^2} - 1) \cosh (\xi_{2n} \frac{r}{a})}$$

the values β_n and g_n and h_n are the coefficients of the expansions

$$(\frac{r}{a})^2 (\ln \frac{r}{a}) = \sum_{n=0}^{\infty} \beta_n C_2 (\xi_{2n} \frac{r}{a})$$

and

$$(\frac{r}{a})^2 = \sum_{n=0}^{\infty} g_n C_2 (\xi_{2n} \frac{r}{a})$$

Similar expressions can be obtained for the force, moments and for the case $\alpha = 3/4$.

D. Special Case: Cylindrical Container with Circular Cross Section

Let the ratio, k = b/a, of the inner to the outer tank diameter approach zero and let α be 1, then we obtain the results which are due to fluid motion in a cylindrical tank with circular cross section. This represents the container with the side wall in the $\varphi = 0$ plane from r = 0 to r = a. Since we restrict ourselves to vibrations in the direction of the x-axis at which the side wall does not disturb the flow field we obtain finally with some limit considerations the velocity potential for translatory excitation in the form:

$$(r,\varphi,z,t) = i\omega x_o e^{i\omega t} a \cos \varphi \{\frac{r}{a} + 2 \sum_{n=0}^{\infty} \frac{J_1 (\epsilon_n \frac{r}{a}) \cosh [\epsilon_n (\frac{z}{a} + \frac{h}{a})]}{(\epsilon_n^2 - 1) J_1 (\epsilon_n) \cosh (\epsilon_n \frac{h}{a}) (\frac{\omega_n^2}{\omega^2} - 1)} \} \quad (49)$$

Here the values ϵ_n are the roots of the equation

$$J_1' = (\epsilon_n) = 0 \qquad n = 0,1,2,.....$$

and the natural circular frequency square is

$$\omega_n^2 = \frac{g}{a} \epsilon_n \tanh (\epsilon_n \frac{h}{a}) \qquad \text{(Figure 5)}$$

where the first values for the ϵ_n values are $\epsilon_0 = 1.84$, $\epsilon_1 = 5.33$, $\epsilon_2 = 8.53$.

As already mentioned, it can be detected that the natural frequency increases with the square root of the longitudinal acceleration g, and that it decreases with increasing tank diameter like

$$1/\sqrt{a}$$

This indicates that the natural frequency of the liquid for large tank diameters is smaller than for small tank diameters. The frequency ratio

$$f_n / \sqrt{g/a}$$

changes only considerably for small fluid height h/a < 1, since for h/a < 1 hyperbolic tangents can be approximated by unity. The free surface displacement is given by

$$\zeta (r,\varphi,t) = \frac{\omega^2 x_o e^{i\omega t} \cos \varphi}{g/a} \{\frac{r}{a} + 2 \sum_{n=1}^{\infty} \frac{J_1 (\epsilon_n \frac{r}{a})}{(\epsilon_n^2 - 1) J_1(\epsilon_n) (\frac{\omega_n^2}{\omega^2} - 1)} \} \tag{50}$$

from which we can conclude that the first term is the displacement with respect to small exciting frequencies. For this, if one neglects the terms with ω^4 compared to those with ω^2 the free fluid surface is a plane proportional to the form r cos φ. With increasing exciting amplitude x_o the surface displacement is increased, while for increasing longitudinal acceleration of the container the disturbances of the free fluid acceleration are slightly decreasing. The fluid force on the tank wall is:

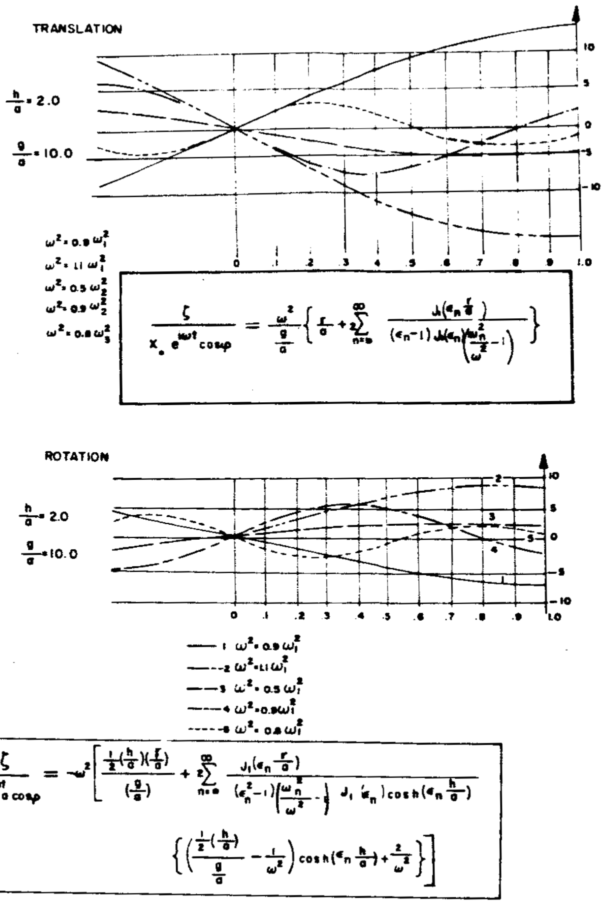

Figure 7. Free Fluid Surface Displacement in a Circular Cylindrical Tank for Various Exciting Frequencies

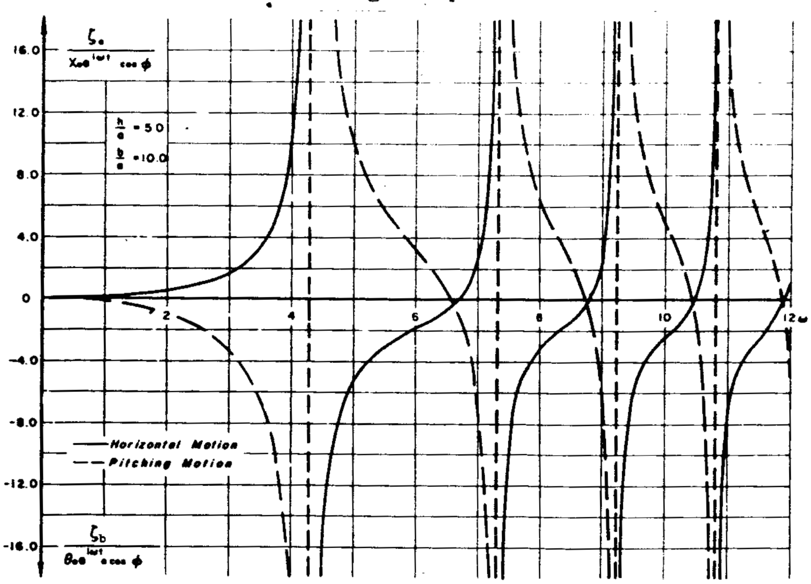

Figure 8. Free Fluid Surface Displacement (Magnification Factor)

$$F_x = m\omega^2 e^{i\omega t} x_o \left\{1 + 2\sum_{n=1}^{\infty} \frac{\tanh\left(\epsilon_n \frac{h}{a}\right)}{\left(\epsilon_n \frac{h}{a}\right)\left(\epsilon_n^2 - 1\right)\left(\frac{\omega_n^2}{\omega^2} - 1\right)}\right\} \tag{51}$$

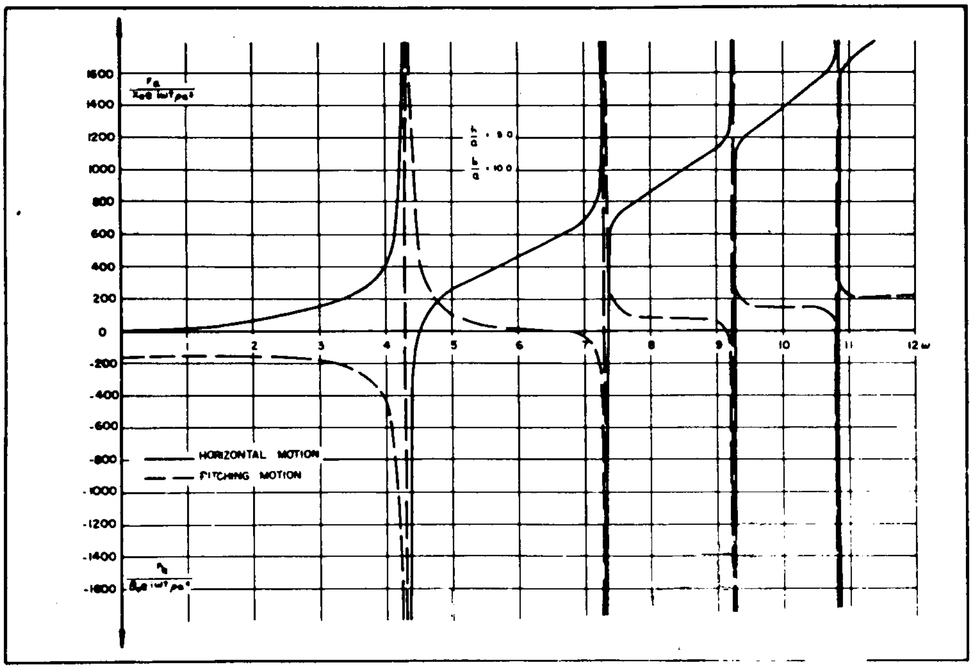

Figure 9. Fluid Forces (Magnification Factor)

where the first term is due to the inertial force. The fluid moment referred to the center of gravity of the undisturbed liquid is for translational oscillation

$$\vec{M}_{y4} = m\omega^2 a x_o e^{i\omega t} \left\{\frac{1}{4\frac{h}{a}} + 2\sum_{n=1}^{\infty} \frac{\frac{1}{2}\tanh\left(\epsilon_n \frac{h}{a}\right) + \frac{1}{\left(\epsilon_n \frac{h}{a}\right)}\left(\frac{2}{\cosh\left(\epsilon_n \frac{h}{a}\right)} - 1\right)}{\epsilon_n\left(\epsilon_n^2 - 1\right)\left(\frac{\omega_n^2}{\omega^2} - 1\right)}\right\} \tag{52}$$

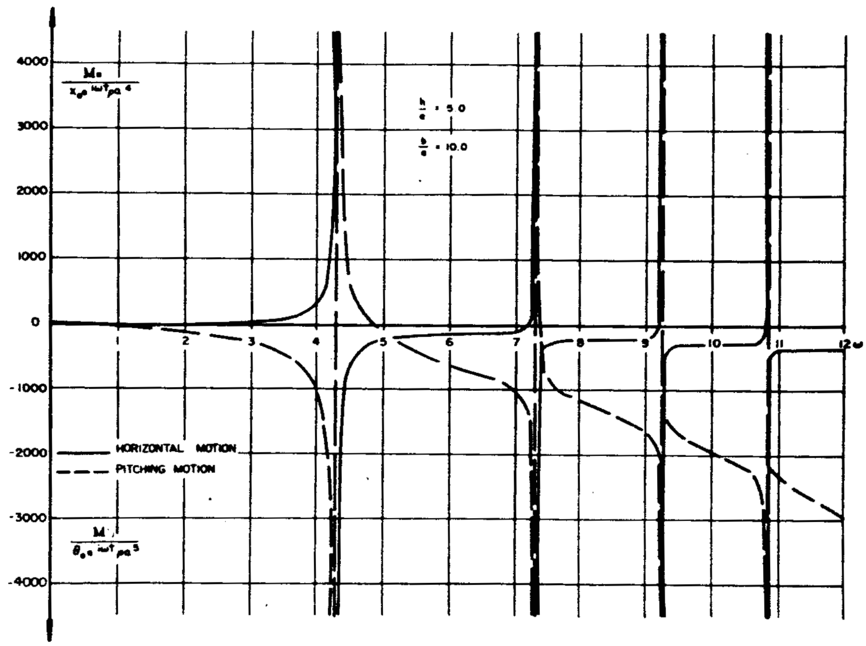

Figure 10. Fluid Moments (Magnification Factor)

The first term in the parentheses is due to the moment of lateral displacement of center of gravity if one considers the free fluid surface as a plane of the form ζ_o r cos φ. It is

$$x_{cg} = \frac{1}{m} \int_0^{2\pi} \int_0^{a} \int_0^{h + \zeta_o r \cos\varphi} \rho_o \, r^2 \cos\varphi \, dr \, d\varphi \, dz, \text{ which is}$$

$$x_{cg} = \frac{a^2 \tan\alpha}{4h} \qquad \text{with } \zeta_o = \tan\alpha = \frac{\ddot{x}}{g}$$

Therefore the moment due to this part is

$$M_y = - mg\, x_{cg} = - \frac{m \ddot{x} a}{4 \frac{h}{a}}$$

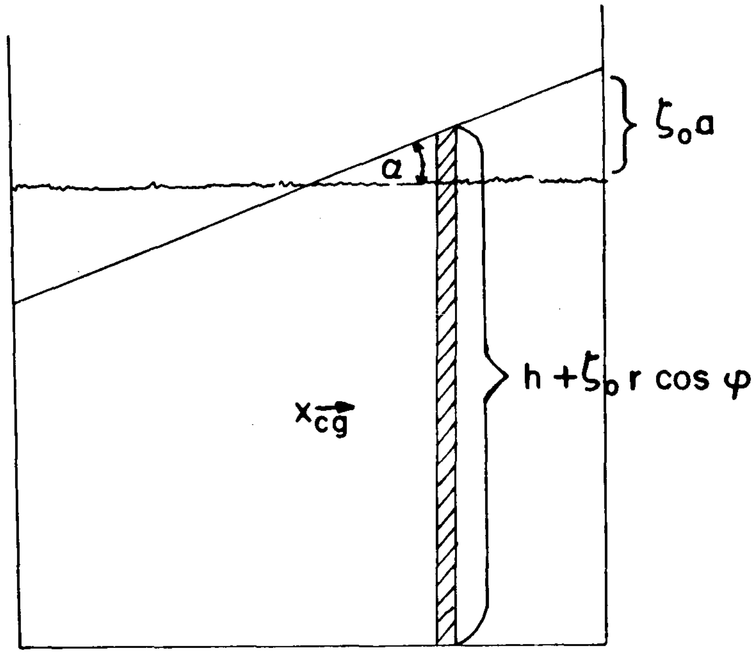

Figure 11. Shift of Center of Gravity During Oscillation

The shifting of the center of gravity in vertical direction (2. order term) need not be considered since all terms of second order originally have been neglected. Similar results can be obtained for any of the tank shapes and has been worked out for cylindrical tank with annular ring cross section [Ref. 1,2] and for a circular cylindrical quarter tank of which a few results will be given in Figures 12 - 18.

The theory, however, does not yield the answer of the liquid for large amplitudes which occur near or at resonance. In the vicinity of the natural frequencies of the liquid (especially the lower frequencies), values appear which represent the important influence on the space vehicle.

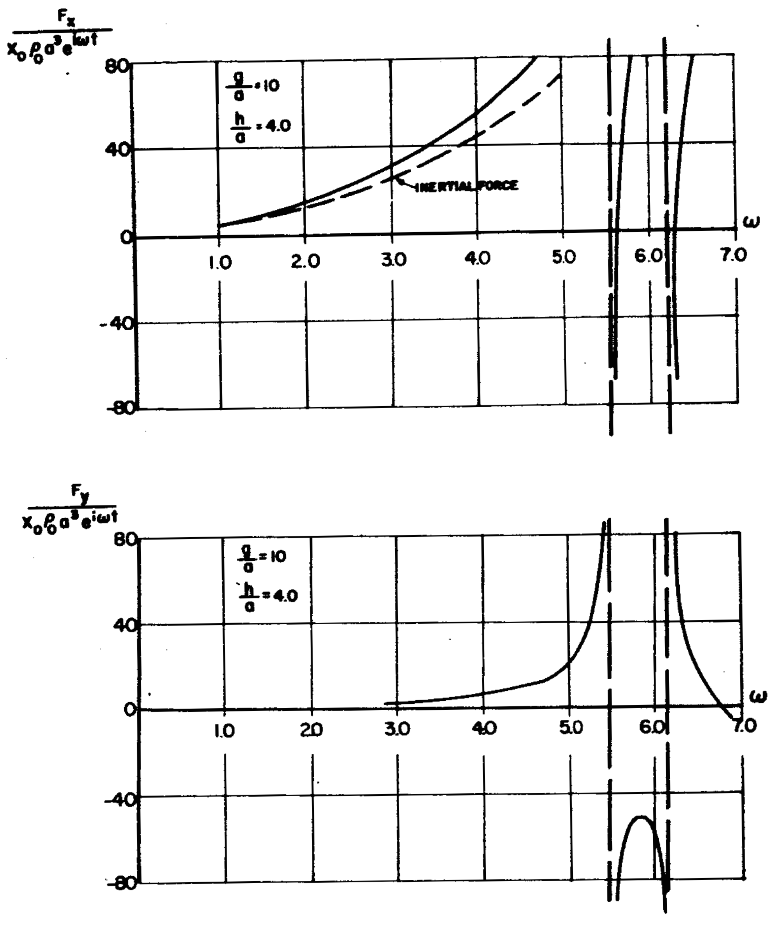

Figure 12. Fluid Force for Translational Excitation in x-Direction of a Liquid in a Quarter Tank

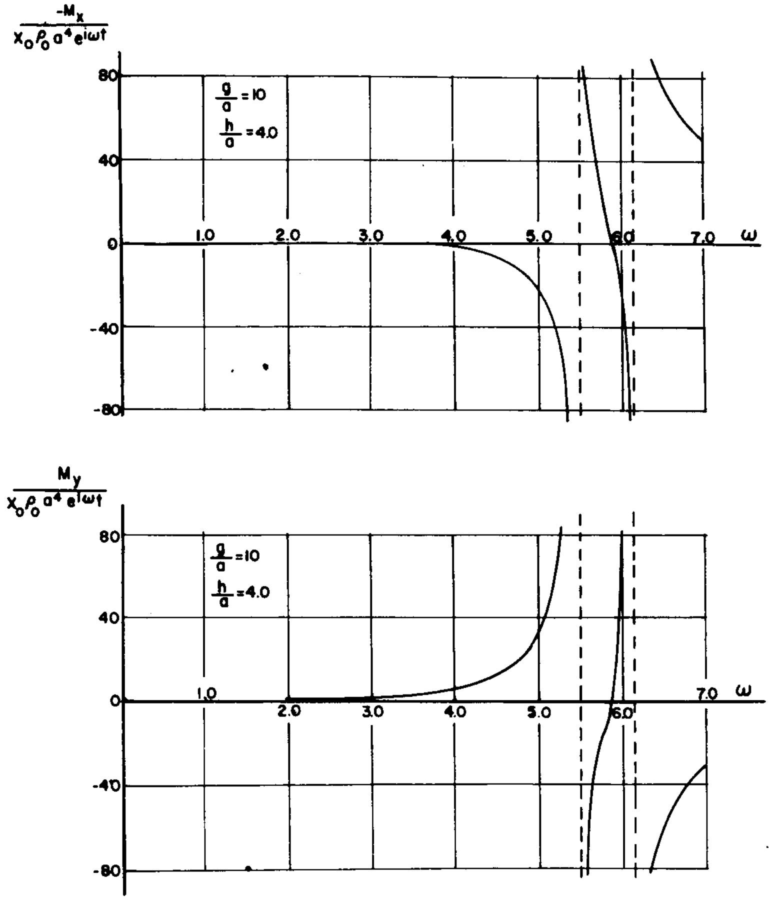

Figure 13. Fluid Moment for Translational Excitation in x-Direction of a Liquid in a Quarter Tank

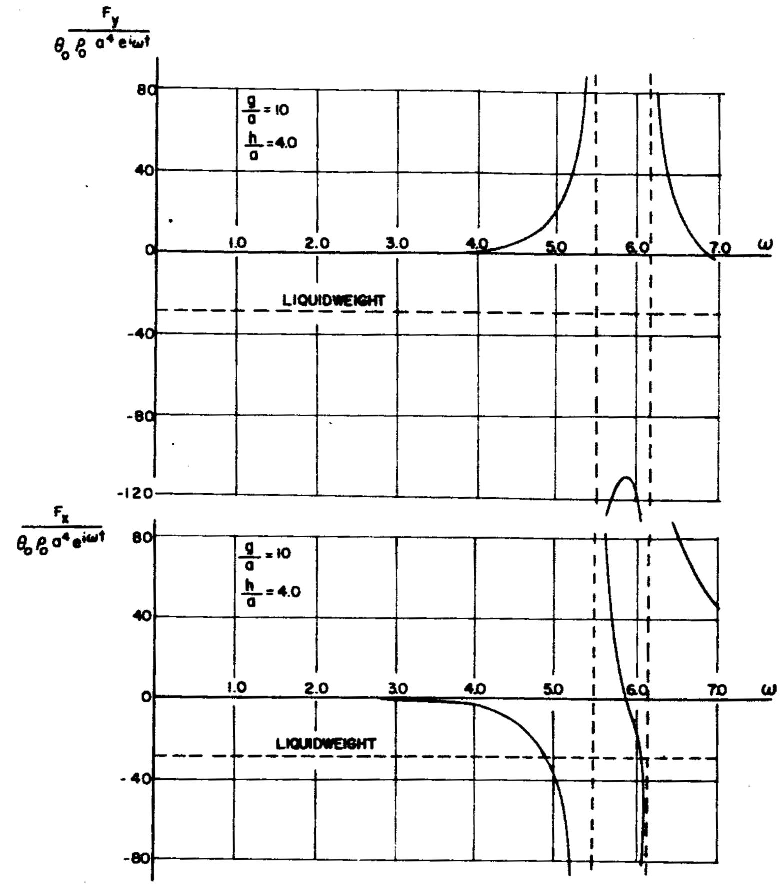

Figure 14. Fluid Force for Rotational Excitation Along the y-Axis in a Quarter Tank

It also gives no answer for the inclusion of mechanical suppression devices or baffles of one type or another. These baffles or at least stiffener rings are almost universally employed to reduce the magnitude of forces and torques from the liquid in the vehicle. This means that, due to the complexity of the fluid flow behavior, mathematical treatment is not possible. Recourse must be made to potential theory with smooth tank walls and experimental investigations from which the damping is obtained and introduced in an equivalent way into the theoretical results. This part will be treated in a later report.

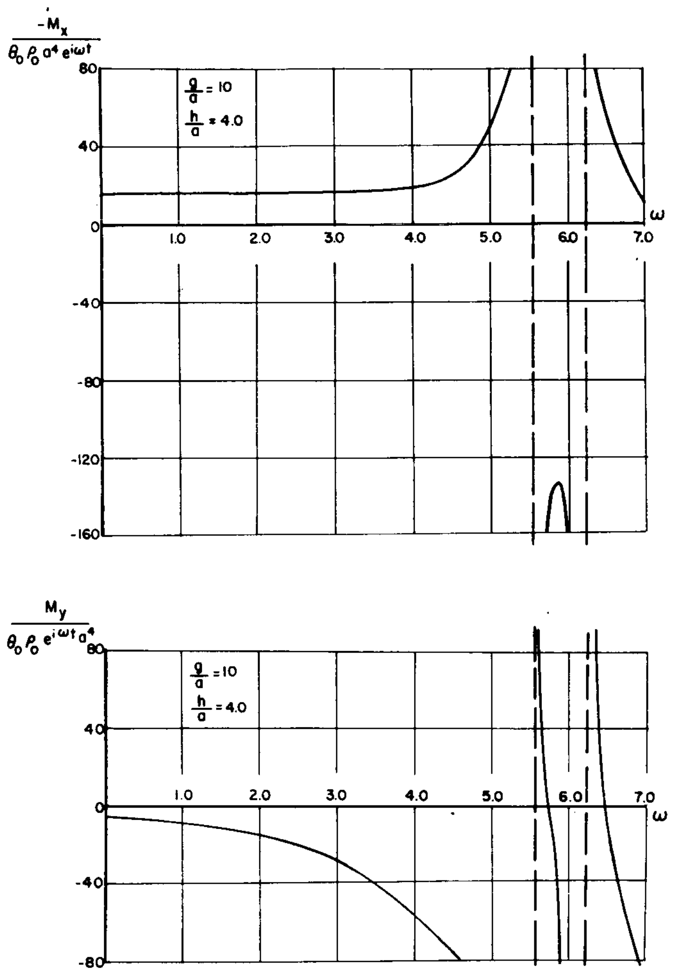

Figure 15. Fluid Moment for Rotational Excitation Along the y-Axis in a Quarter Tank

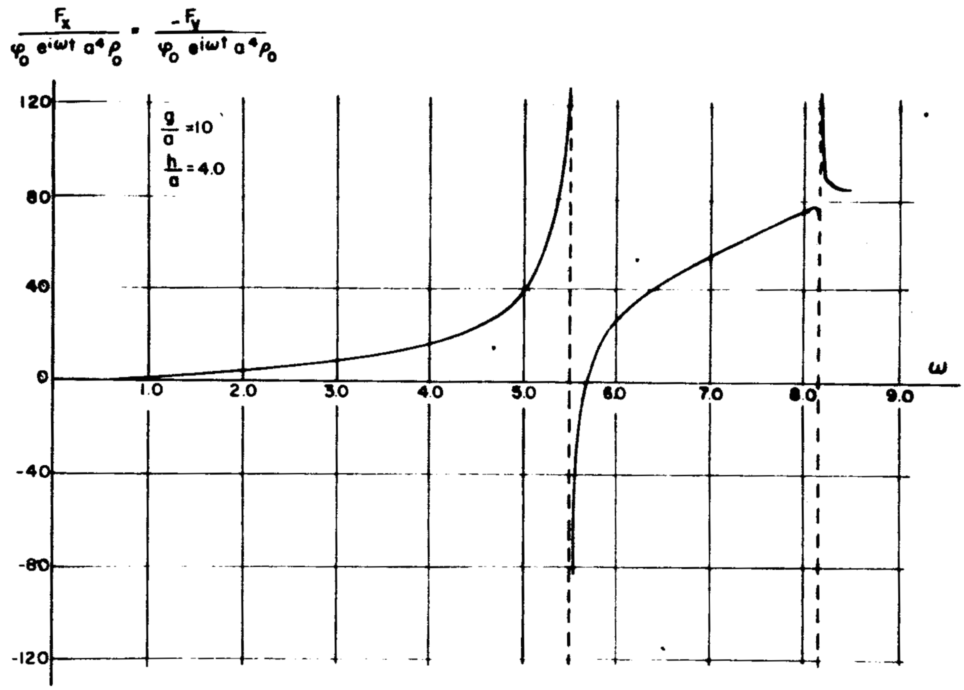

Figure 16. Fluid Forces for Roll Excitation of a Quarter Tank

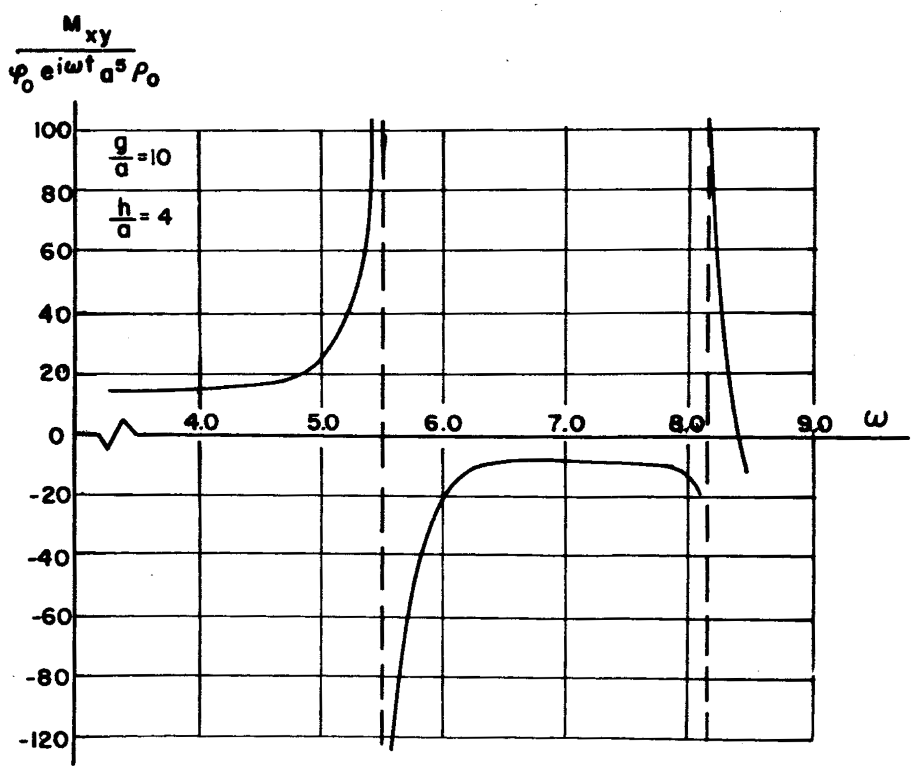

Figure 17. Fluid Moment for Roll Excitation of a Quarter Tank

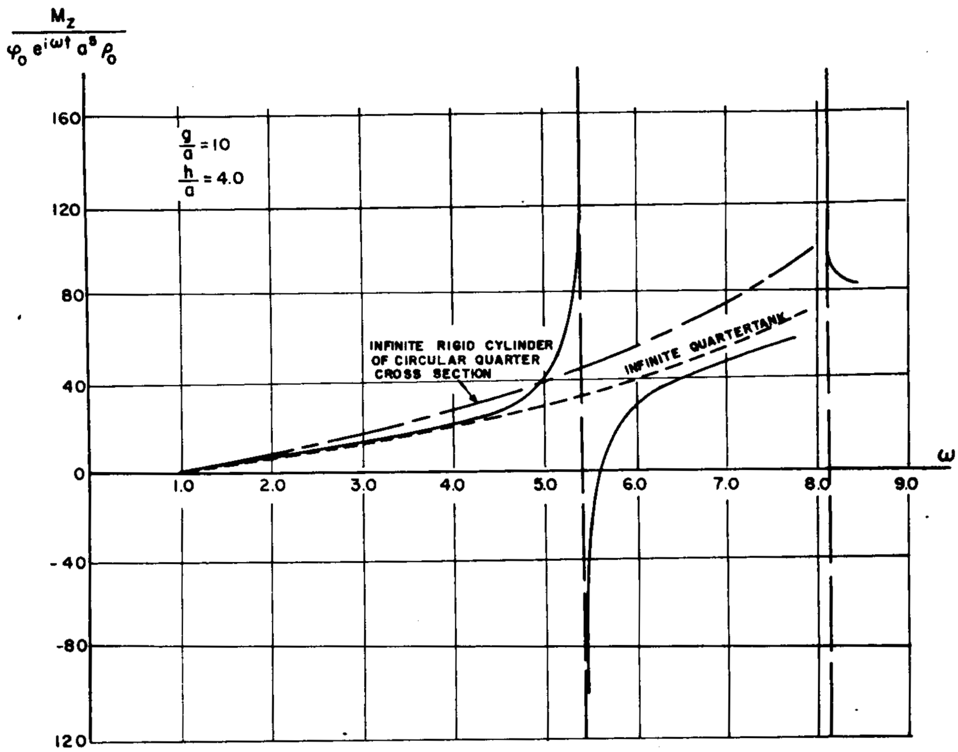

Figure 18. Fluid Moment for Roll Excitation of a Quarter Tank

V. CONCLUSION

The natural frequency of a liquid in a cylindrical container with circular annular sector cross sections increases proportionally to the square root of the longitudinal acceleration. This means that, during the flight of a space vehicle, which increases its acceleration during flight time, the natural frequency increases. Only during the burn out period the natural frequency of the propellant is decreasing again, since the fluid height influence [tank $(\xi_n h/a)$] overcomes the influence of the acceleration g. If one chooses a larger tank diameter, then the natural frequency of the propellant is lower due to its being proportional to $1/\sqrt{a}$. This indicates, that the natural frequency in large space vehicles with large tank diameters is very low, thus being closer to the control frequency. This is a very unfavorable situation, since one wants the natural frequency of the liquid as far above the control frequency as possible. An increase in the natural frequency of the liquid can also be obtained by tank geometry. Subdividing a tank therefore e.g. into quarter tanks increases the natural frequency due to the roots of

$$\Delta_{\frac{m}{2\alpha}}(\xi) = 0.$$

As already mentioned, it could be seen, that for a tank with circular cross section the roots of $\Delta_1 \equiv J_1'(\epsilon) = 0$ were $\epsilon_0 = 1.84$, $\epsilon_1 = 5.33$, etc. For a quarter tank arrangement the roots are obtained from $J_{2m}'(\epsilon) = 0$ and we obtain $\epsilon_{00} \approx 3.832$, $\epsilon_{10} \approx 3.054$, which indicates that the lowest natural frequency of the liquid is increased by about a factor of 1.4.

It was seen, that once the velocity potential has been determined, the pressure and velocity distribution, the free fluid surface displacement, as well as the forces and torques of the liquid could be determined by differeniations and integrations with respect to the spacial and time coordinates. These results have at the resonance frequencies of the liquid singularities, which can be eliminated by the introduction of damping in the resonance terms. This will be treated in a later report.

TABLE 1

Roots of the Determinate

$$\Delta_1 (\xi) = 0$$

n \ k	0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	1.8412	1.8035	1.7051	1.5821	1.4618	1.3547	1.2621	1.2824	1.1134	1.0532
1	5.3314	5.1371	4.9609	5.1374	5.6592	6.5649	8.0411	10.5919	15.7781	31.4470
2	8.5363	8.1992	8.4331	9.3084	10.6834	12.7066	15.8013	21.0041	31.4513	62.8481
3	11.7060	11.3588	12.1650	13.6837	15.8481	18.9427	23.6239	31.4558	47.1504	94.2645
4	14.8636	14.6344	15.9932	18.1159	21.0488	25.2025	31.4632	41.9190	62.8510	125.6750
5	18.0155	17.9864	19.8616	22.5707	26.2641	31.4721	39.3076	52.3845	78.5549	157.0880
6	21.1644	21.3837	23.7502	27.0369	31.4859	37.7459	47.1552	62.8523	94.2601	188.5022
7	24.3114	24.8081	27.6498	31.5092	36.7119	44.0223	55.0047	73.3212	109.9662	219.9170
8	27.4575	28.2497	31.5563	35.9855	41.9403	50.3005	62.8553	83.7910	125.6728	251.3320
9	30.6019	31.7027	35.4675	40.4643	47.1704	56.5797	70.7066	94.2612	141.3797	282.7473

TABLE 2

Roots of $J'_{2m}(\epsilon_{nm}) = 0$

n \ m	0	1	2	3	4	5	6	7	8	9
0	3.832	3.054	5.318	8.105	9.648	11.716	13.821	15.917	18.104	20.189
1	7.016	6.706	9.282	11.735	14.115	16.448	18.745	21.015	23.264	25.495
2	10.173	9.969	12.682	15.268	17.774	20.223	22.629	25.002	27.347	29.670
3	13.324	13.170	15.964	18.637	21.229	23.761	26.246	28.694	31.112	33.504
4	16.471	16.348	19.196	21.932	24.587	27.182	29.729	32.237	34.712	37.160
5	19.616	19.513	22.401	25.184	27.889	30.535	33.131	35.689	38.212	40.707
6	22.760	22.672	25.590	28.410	31.155	33.842	36.481	39.079	41.643	44.178
7	25.904	25.826	28.768	31.618	34.397	37.118	39.792	42.426	45.052	47.595
8	29.047	28.978	31.939	34.813	37.620	40.371	43.075	45.740	48.371	50.971
9	32.189	32.127	35.104	38.000	40.830	43.607	46.338	49.030	51.687	54.315

APPENDIX

A. Roots of Certain Bessel Functions. For the previous results the roots of

$$\Delta_{\frac{m}{2\alpha}}(\xi) = \begin{vmatrix} J'_{\frac{m}{2\alpha}}(\xi) & Y'_{\frac{m}{2\alpha}}(\xi) \\ J'_{\frac{m}{2\alpha}}(k\xi) & Y'_{\frac{m}{2\alpha}}(k\xi) \end{vmatrix} = 0$$

have to be determined for m = 0,1,2,.... and arbitrary $0 \leq k < 1$. For most of these roots J. McMahon represented asymptotic expansions (Ref. 5). The smallest root, however, was not known until H. Buchholz pointed out its existence (Ref. 6). In Ref. 7 D. Kirkham gave the roots of the above equation in a graphical way for $m/2\alpha = 0, 1, 2, 3, 4$. In the numerical evaluation of fluid oscillations in containers with an annular circular cross section, the roots of $\Delta_1(\xi) = 0$ have to be known for various diameter ratios k. These roots have been determined numerically and are given in Table 1. The roots of the equation $J'_{2m}(\epsilon) = 0$ which appear in a cylindrical container of circular quarter cross section represented in Table 2.

B. Representation of a Function in Bessel-Fourier-Series

The determinant $C_{\frac{m}{2\alpha}}$ is

$$C_{\frac{m}{2\alpha}}(\lambda_{mn} r) = \begin{vmatrix} J_{\frac{m}{2\alpha}}(\lambda_{mn} r) & Y_{\frac{m}{2\alpha}}(\lambda_{mn} r) \\ J'_{\frac{m}{2\alpha}}(\lambda_{mn} a) & Y'_{\frac{m}{2\alpha}}(\lambda_{mn} a) \end{vmatrix}$$

Its derivative is

$$C'_{\frac{m}{2\alpha}}(\lambda_{mn} r) = \begin{vmatrix} J'_{\frac{m}{2\alpha}}(\lambda_{mn} r) & Y'_{\frac{m}{2\alpha}}(\lambda_{mn} r) \\ J'_{\frac{m}{2\alpha}}(\lambda_{mn} a) & Y'_{\frac{m}{2\alpha}}(\lambda_{mn} a) \end{vmatrix}$$

which vanishes for r = a and r = b that is

$$C'_{\frac{m}{2\alpha}}(\lambda_{mn} a) = C'_{\frac{m}{2\alpha}}(\lambda_{mn} b) = 0$$

for r = a the derivative of $C_{\frac{m}{2\alpha}}$

vanishes identically, while for r = b the roots $(\xi_{mn}) = (\lambda_{mn} a)$ make it vanish.

A function f (r) which is piecewise regular in the interval

$b \leq r \leq a$,

satisfies the Dirichlet condition, can be expanded into a Bessel-Fourier series of the form

$$f(r) = \sum_{n=0}^{\infty} b_{mn}^{(f)} \; C_{\frac{m}{2\alpha}}(\lambda_{mn} r) \qquad (m = 0,1,2,\ldots..)$$

the unknown coefficients of the expansion will be determined by multiplying both sides of the equation with $r\, C_{\frac{m}{2\alpha}}(\lambda_{mp} r)$

and integrating from r = b to r = a. Here λ_{mp} and λ_{mn} are different roots of the determinant $\Delta_{\frac{m}{2\alpha}} = 0$. It is

$$\sum_{n=0}^{\infty} b_{mn} \int_a^b r\, C_{\frac{m}{2\alpha}}(\lambda_{mn} r)\, C_{\frac{m}{2\alpha}}(\lambda_{mp} r)\, dr = \int_b^a r\, f(r)\, C_{\frac{m}{2\alpha}}(\lambda_{mp} r)\, dr$$

With the integral of Lommel we obtain

$$(\lambda_{mn}^2 - \lambda_{mp}^2) \int r\, C_{\frac{m}{2\alpha}}(\lambda_{mn} r)\, C_{\frac{m}{2\alpha}}(\lambda_{mp} r)\, dr = r \left\{ C_{\frac{m}{2\alpha}}(\lambda_{mn} r) \frac{dC_{\frac{m}{2\alpha}}(\lambda_{mp} r)}{dr} - C_{\frac{m}{2\alpha}}(\lambda_{mp} r) \frac{dC_{\frac{m}{2\alpha}}(\lambda_{mn} r)}{dr} \right\} \quad (n \neq p)$$

and the integral on the left hand side is

$$\int r\, C_{\frac{m}{2\alpha}}(\lambda_{mn} r)\, C_{\frac{m}{2\alpha}}(\lambda_{mp} r)\, dr = \frac{r}{(\lambda_{mn}^2 - \lambda_{mp}^2)} \{\lambda_{mp}\, C_{\frac{m}{2\alpha}}(\lambda_{mn} r)\, C'_{\frac{m}{2\alpha}}(\lambda_{mp} r) - \lambda_{mn}\, C_{\frac{m}{2\alpha}}(\lambda_{mp} r)\, C'_{\frac{m}{2\alpha}}(\lambda_{mn} r)\} \quad (p \neq n) \tag{F}$$

The integral is zero, if one of the following conditions is satisfied.

$$1.\quad C_{\frac{m}{2\alpha}}(\lambda_{mn} a) = C_{\frac{m}{2\alpha}}(\lambda_{mp} a) = C_{\frac{m}{2\alpha}}(\lambda_{mn} b) = C_{\frac{m}{2\alpha}}(\lambda_{mp} b) = 0$$

$$2.\quad C'_{\frac{m}{2\alpha}}(\lambda_{mn} a) = C'_{\frac{m}{2\alpha}}(\lambda_{mp} a) = C'_{\frac{m}{2\alpha}}(\lambda_{mn} b) = C'_{\frac{m}{2\alpha}}(\lambda_{mp} b) = 0 \tag{G}$$

$$3.\quad \lambda_{mp}\, C_{\frac{m}{2\alpha}}(\lambda_{mn} a)\, C'_{\frac{m}{2\alpha}}(\lambda_{mp} a) = \lambda_{mn}\, C_{\frac{m}{2\alpha}}(\lambda_{mp} a)\, C'_{\frac{m}{2\alpha}}(\lambda_{mn} a) \text{ and}$$

$$\lambda_{mp}\, C_{\frac{m}{2\alpha}}(\lambda_{mn} b)\, C'_{\frac{m}{2\alpha}}(\lambda_{mp} b) = \lambda_{mn}\, C_{\frac{m}{2\alpha}}(\lambda_{mp} b)\, C'_{\frac{m}{2\alpha}}(\lambda_{mn} b)$$

In the here treated $C'_{\frac{m}{2\alpha}}$ cases of fluid oscillations the second boundary condition is satisfied, since $\lambda_{mn} a$ and $\lambda_{mn} b$ are roots of the equation $\Delta_{\frac{m}{2\alpha}} = 0$.

Those terms for which $\lambda_{mn} \neq \lambda_{mp}$ vanish and one obtains for the coefficients

$$b_{mn}^{(f)} = \frac{\int_b^a r\, f(r)\, C_{\frac{m}{2\alpha}}(\lambda_{mn} r)\, dr}{\int_b^a r\, C^2_{\frac{m}{2\alpha}}(\lambda_{mn} r)\, dr} \tag{H}$$

For $p = n$ the equation (F) will be an indeterminate form, which will be treated with Taylor expansion or the rule of L'Hospital, and is with the Bessel differential equation for $C_{\frac{m}{2\alpha}}$

$$\int r\, C^2_{\frac{m}{2\alpha}} (\lambda_{mn} r)\, dr = \frac{r^2}{2} \{C^2_{\frac{m}{2\alpha}} (\lambda_{mn} r) [1 - \frac{m^2}{4\alpha^2 \lambda^2_{mn} r^2}] + C'^2_{\frac{m}{2\alpha}} (\lambda_{mn} r)\} \qquad (I)$$

We thus obtain in the interval $b \leq r \leq a$

$$\int_b^a r\, C^2_{\frac{m}{2\alpha}} (\lambda_{mn} r)\, dr = \frac{a^2}{2} [C^2_{\frac{m}{2\alpha}} (\lambda_{mn} a) (1 - \frac{m^2}{4\alpha^2 \lambda^2_{mn} a^2}) + C'^2_{\frac{m}{2\alpha}} (\lambda_{mn} a)] - \frac{b^2}{2} [C^2_{\frac{m}{2\alpha}} (\lambda_{mn} b) (1 - \frac{m^2}{4\alpha^2 \lambda^2_{mn} b^2}) + C'^2_{\frac{m}{2\alpha}} (\lambda_{mn} b)]. \qquad (J)$$

which is due to the boundary conditions

$$\int_b^a r\, C^2_{\frac{m}{2\alpha}} (\xi_{mn} \frac{r}{a})\, dr = \frac{a^2}{2\xi^2_{mn}} [\frac{4}{\pi^2 \xi^2_{mn}} (\xi^2_{mn} - \frac{m^2}{4\alpha^2}) - C^2_{\frac{m}{2\alpha}} (k\xi_{mn}) (k^2\xi^2_{mn} - \frac{m^2}{4\alpha^2})] \qquad (K)$$

Here $C_{\frac{m}{2\alpha}} (\xi_{mn}) = \frac{2}{\pi\xi_{mn}}$ is the Wronskian determinant. The coefficient $b^{(f)}_{mn}$ of the Bessel-Fourier expansion can be determined from

$$b^{(f)}_{mn} = \frac{2\xi^2_{mn} \int_b^a r\, f(r)\, C_{\frac{m}{2\alpha}} (\xi_{mn} \frac{r}{a})\, dr}{a^2 [\frac{4}{\pi^2 \xi^2_{mn}} (\xi^2_{mn} - \frac{m^2}{4\alpha^2}) - C^2_{\frac{m}{2\alpha}} (k\xi_{mn}) (k^2 \xi^2_{mn} - \frac{m^2}{4\alpha^2})]} \qquad (L)$$

The problem that remains is the solution of the $\int_b^a r\, f(r)\, C_{\frac{m}{2\alpha}} (\xi_{mn} \frac{r}{a})\, dr$.

Most of the integrals in the previous treatment are of the form

$$\int z^{\kappa}\, C_{\nu}\,(z)\,dz\,..$$

These can be obtained with the help of the Lommel function $S_{\kappa\nu}\,(z)$ or by integration of the series expansion of the integrals.

$$\int z^{\kappa}\, C_{\frac{m}{2\alpha}}\,(z)\,dz = Y'_{\frac{m}{2\alpha}}\,(\xi_{mn}) \int z^{\kappa}\, J_{\frac{m}{2\alpha}}\,(z)\,dz - J'_{\frac{m}{2\alpha}}\,(\xi_{mn}) \int z^{\kappa}\, Y_{\frac{m}{2\alpha}}\,(z)\,dz \tag{M}$$

Integrating the first integral term by term and collecting terms of $J_{\nu\ +\ 2\mu\ +\ 1}$, one obtains

$$\int z^{\kappa}\, J_{\nu}\,(z)\,dz = \frac{z^{\kappa}\,\Gamma\,(\frac{\kappa+\nu+1}{2})}{\Gamma\,(\frac{\nu-\kappa+1}{2})} \sum_{\mu=0}^{\infty} \frac{(\nu+2\mu+1)\ \Gamma\,(\frac{\kappa+\kappa+1}{2}+\mu)}{\Gamma\,(\frac{\nu+\kappa+3}{2}+\mu)}\ J_{\nu+2\mu+1} \tag{N}$$

where Re $(\kappa+\mu+1)$ must be > 0 if one integrates from $z = 0$ on.

The second integral is obtained by term wise integration of the series expansion of the Bessel function of second kind.

It is for ($\frac{m}{2\alpha}$ integer)

$$\int z^{\kappa}\, Y_{\frac{m}{2\alpha}}\,(z)\,dz) = -\frac{z^{\kappa+1}}{\pi} \sum_{\mu=0}^{\frac{m}{2\alpha}-1} \frac{(\frac{m}{2\alpha}-\mu-1)!\ \ (\frac{z}{2})^{2\mu-\frac{m}{2\alpha}}}{\mu!\ (\kappa+2\mu-\frac{m}{2\alpha}+1)} +$$

$$\frac{2z^{\kappa+1}}{\pi} \sum_{\mu=0}^{\infty} \frac{(-1)^{\mu}\ (\frac{z}{2})^{\frac{m}{2\alpha}+2\mu}}{\mu\,!\,(\frac{m}{2\alpha}+\mu)!\ \ (\frac{m}{2\alpha}+\kappa+2\mu+1)} \cdot \{\ln\frac{z}{2} - \frac{1}{2}\,\psi\,(\mu+1) -$$

$$-\frac{1}{2}\,\psi\,(\mu+\frac{m}{2\alpha}+1)\} - \frac{2z^{\kappa+1}}{\pi} \sum_{\mu=0}^{\infty} \frac{(-1)^{\mu}\ (\frac{z}{2})^{\frac{m}{2\alpha}+2\mu}}{\mu\,!\,(\frac{m}{2\alpha}+\mu)\,!\,(\frac{m}{2\alpha}+\kappa+2\mu+1)^2} \tag{0}$$

where $\psi(z)$ represents the logarithmic derivative of the Gamma function

$$\psi(z) = \frac{d\,(\ln\Gamma(z))}{dz} = -\gamma + (z-1)\sum_{\lambda=0}^{\infty}\frac{1}{(\lambda+1)(z-\lambda)} \tag{P}$$

and γ is the Euler constant. With these results we obtain the integrals as mentioned in the text.

$$\int_b^a r^2 C_{\frac{m}{2\alpha}}\left(\xi_{mn}\frac{r}{a}\right) dr = a^3 N_2^{\left(\frac{m}{2\alpha}\right)}(\xi_{mn})$$

$$\int_b^a r\, C_{\frac{2m-1}{2\alpha}}\left(\xi_{mn}\frac{r}{a}\right) dr = a^2 N_1^{\frac{2m-1}{2\alpha}}(\xi_{(2m-1),n}) \tag{Q}$$

$$\int_b^a C_{\frac{m}{2\alpha}}\left(\xi_{mn}\frac{r}{a}\right) dr = a\, N_o^{\left(\frac{m}{2\alpha}\right)}(\xi_{mn})$$

Similar results can be obtained for the integrals in roll oscillation. It may be mentioned here that some of the integrals in which κ is $1-\nu$ or $\nu+1$ can be obtained from the recursion formulas

$$\int z^{1-\nu} C_\nu(z)\, dz = -z^{1-\nu} C_{\nu-1}(z) = 0\, z^{1-\nu} C'_\nu(z) - z^{-\nu}\nu\, C_\nu(z)$$

$$\int z^{\nu+1} C_\nu(z)\, dz = z^{\nu+1} C_{\nu+1}(z) = \nu z^{\nu} C_\nu(z) - z^{\nu+1} C'_\nu(z)$$

Integrals which contain in the function $f(r)$ a $\ln(r/a)$ can be determined in the same way as previous if one performs first an integration by parts.

C. Limit Considerations for $k \to 0$

The previous results can be applied for cylindrical tanks with circular cross section by letting $k \to 0$. The zeros of the determinant

Δ_ν. $\left(\nu = \frac{m}{2\alpha}\right)$

$$\Delta_\nu(\xi) = \begin{vmatrix} J'_\nu(\xi) & Y'_\nu(\xi) \\ J'_\nu(k\xi) & Y'_\nu(k\xi) \end{vmatrix} = 0$$

approaches for $k \to 0$ the value $J'_{\frac{m}{2\alpha}}$. This is due to the fact that

$$J_\nu(x) = \frac{x^\nu}{2^\nu \Gamma(\nu+1)}$$

for small x and

$$Y_\nu(x) \approx \frac{-2^\nu \Gamma(\nu)}{\pi x^\nu}$$

for $\nu > 0$ and small x. Instead of the value

$$C_{\frac{m}{2\alpha}}\left(\xi_{mn}\frac{r}{a}\right)$$

in a ring sector tank the values of $J_{\frac{m}{2\alpha}}\left(\xi_{mn}\frac{r}{a}\right)$ has to be taken for a container of circular sector cross section.

With (N) we obtain the values L_0, L_1, L_2 for the sector tank.

$$L_0^{(\frac{m}{2\alpha})}(\epsilon_{mn}) = \frac{2}{\epsilon_{mn}}\sum_{\mu=0}^{\infty} J_{2\mu+\frac{m}{2\alpha}+1}(\epsilon_{mn}) \qquad \left(\mathrm{Re}\,\frac{m}{2\alpha} > -1\right)$$

$$L_1^{(\frac{2m-1}{2\alpha})}(\epsilon_{mn}) = \frac{2m-1}{4\alpha\,\epsilon_{2m-1,n}}\sum_{\mu=0}^{\infty}\frac{\left(\frac{2m-1}{2\alpha}+2\mu+1\right)}{\left(\frac{2m-1}{4\alpha}+\mu\right)\left(\frac{2m-1}{4\alpha}+\mu+1\right)} J_{\frac{2m-1}{2\alpha}+2\mu+1}(\epsilon_{2m-1,n})$$

$$L_2^{(\frac{m}{2\alpha})}(\epsilon_{mn}) = \frac{\Gamma\left(\frac{m}{4\alpha}+\frac{3}{2}\right)}{\epsilon_{mn}\,\Gamma\left(\frac{m}{4\alpha}-\frac{1}{2}\right)}\sum_{\mu=0}^{\infty}\frac{\left(\frac{m}{2\alpha}+2\mu+1\right)\Gamma\left(\frac{m}{2\alpha}+\mu-\frac{1}{2}\right)}{\Gamma\left(\frac{m}{4\alpha}+\mu+\frac{5}{2}\right)} J_{\frac{m}{2\alpha}+2\mu+1}(\epsilon_{mn})$$

The other values can be obtained in a similar way.

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THEORY OF FLUID OSCILLATIONS IN PARTIALLY FILLED CYLINDRICAL CONTAINERS

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